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Author(s): Paul Milgrom and Chris Shannon

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MONOTONE COMPARATIVE STATICS

BY PAUL MILGROM AND CHRIS SHANNON[1]

We derive a necessary and sufficient condition for the solution set of an optimization problem to be monotonic in the parameters of the problem. In addition, we develop practical methods for checking the condition and demonstrate its applications to the classical theories of the competitive firm, the monopolist, the Bertrand oligopolist, consumer and growth theory, game theory, and general equilibrium analysis.

KEYWORDS: Supermodular, quasisupermodular, single crossing condition, comparative statics, strategic complements.

1. INTRODUCTION

THE MOST COMMON METHODS for comparative statics analyses in modern economics are based on applying the implicit function theorem to first-order conditions or on exploiting the identities of duality theory. Of course, in order to apply these methods, certain assumptions must be satisfied. The most common such assumptions concern the convexity of preferred sets or constraint sets, the smoothness of indifference curves or boundaries of constraint sets, derivative conditions such as Inada conditions that ensure interior solutions, strict second derivative conditions or conditions regarding the positive or negative definiteness of the Hessian, and the linearity of budget sets or objective functions.

It is important to recognize that the *only* role these assumptions play is as servants to a method. No combination of these assumptions could ever be either necessary or sufficient for any nontrivial conclusion about the direction of change of the endogenous choice variables in response to changes in exogenous parameters. Indeed, consider any parameterized family of optimization problems with choice variables x and parameters t. If some combination of these assumptions holds for the pair of variables (x, t), then the same combination of assumptions also holds for the variables $(x, -t)$. For example, given the constraint $(x, t) \in C$, an equivalent formulation of this constraint is the requirement that $(x, -t) \in \hat{C}$, where $\hat{C} \equiv \{(x, -t) | (x, t) \in C\}$. Then the set C will be convex, have smooth (or linear) boundaries, be open and so on if and only if the set $\hat{C}$ has the same property; consequently, if such conditions were sufficient to imply that the optimum x^* is a nondecreasing function of t over some range of parameter values, they would also imply that x^* is a nondecreasing function of $-t$ over the corresponding range, which is possible only in the trivial case.

It is even clearer that these conditions couldn't be necessary for any meaningful comparative statics conclusions. To see this, suppose that g is any discontinuous function that is increasing and has an increasing inverse, and let $y = g(x)$.

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Then the parameterized problem

Maximize $f(x;t)$ subject to $x \in S$

is equivalent to the problem

Maximize $h(y;t)$ subject to $y \in g(S)$

where $h(y;t) = f(g^{-1}(y);t)$. Since g is an order-preserving transformation, the solution $x^*(t)$ is nondecreasing if and only if $y^*(t) = g(x^*(t))$ is nondecreasing. However, even if the original problem were smooth or linear or convex in the choice variable x, the transformed problem is certainly not, and hence the comparative statics conclusions are not predicated on these types of assumptions.

In this paper, we develop a theory and methods for comparative statics analysis using only conditions that are *ordinal*, that is, independent of order-preserving transformations. We identify necessary and sufficient conditions for monotone comparative statics both for individual optimization problems and for certain families of problems. Because our analysis depends only on the order structure of the problems, it is equally effective for both convex and nonconvex problems.

The remainder of the paper is organized as follows. In Section 2, we introduce the relevant mathematical structure, which uses only the order properties of the parameter set and the set of decision variables. There we present the necessary and sufficient condition for monotone comparative statics and relate it to the "Spence-Mirrlees" single crossing condition of incentive theory and information economics. Section 3 contains characterizations of our conditions and also introduces the key applications-oriented theorems, which give necessary and sufficient conditions for monotone comparative statics to hold in each of a parameterized family of problems. Section 4 then illustrates how the methods can be applied to both old and new problems of microeconomics. In Section 5, we apply our conditions to extend the theory of games with strategic complementarities and to analyze general equilibrium models with gross substitutes in demand. Concluding remarks are given in Section 6, followed by an Appendix devoted to studying the nonemptiness and structure of the set of optima.

2. THE THEORY OF MONOTONE COMPARATIVE STATICS

Statements about monotone comparative statics are fundamentally statements about order—they are statements of the form that an increase in some variable leads to increases in other variables—so the machinery needed to develop a general theory of monotone comparative statics is order-theoretical, that is, *lattice theory*.

Let X be a partially ordered set, with the transitive, reflexive, antisymmetric order relation $\geqslant$.[2] For x and y elements of X, let $x \vee y$ denote the least upper

[2] Recall that an order relation is *reflexive* if $x \geqslant x$ for every $x \in X$, and *antisymmetric* if $x \geqslant y$ and $y \geqslant x$ implies that $x = y$.

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bound, or *join*, of x and y in X, if it exists, and let $x \wedge y$ denote the greatest lower bound, or *meet* of x and y in X, if it exists. The set X is a *lattice* if for every pair of elements x and y in X, the join $x \vee y$ and meet $x \wedge y$ do exist as elements of X. Similarly, a subset S of X is a *sublattice* of X if S is closed under the operations meet and join. Finally, a sublattice S of X is *complete* if for every nonempty subset S' of S, $\inf(S')$ and $\sup(S')$ both exist and are elements of S.[3]

It is important to notice that all of these definitions rely on the order $\geqslant$ on the underlying set X. In many applications, X will be $\mathbb{R}^n$ with the component-wise order. In those cases, $x \wedge y$ denotes the component-wise minimum of x and y and $x \vee y$ denotes the component-wise maximum. Furthermore, when X is $\mathbb{R}^n$, a bounded sublattice S of X is complete if and only if it is a compact set in the standard Euclidean topology. A simple example of a set which is not a lattice in the component-wise order is $\{(x_1, x_2) | x_1 + x_2 = 1\}$. However, this same set *is* a lattice in the order in which $x \geqslant y$ if $x_1 \geqslant y_1$ and $x_2 \leqslant y_2$. When the lattice X is a product of lattices, say $A \times B$, then our default specification of order on X is to use the product order, so that $(a, b) \geqslant (a', b')$ if $a \geqslant a'$ in A and $b \geqslant b'$ in B.

Monotone comparative statics requires an order both on the set of constraints and on the set of maximizers. The one we will use is the *strong set order* $\leqslant_s$ introduced by Veinott (1989). For X a lattice with the given relation $\geqslant$, with Z and Y elements of the power set $P(X)$, we say that $Z \leqslant_s Y$, read "Y is higher than Z", if for every $z \in Z$ and $y \in Y$, $z \wedge y \in Z$ and $z \vee y \in Y$. Given a partially ordered set T, we say that a set-valued function M: $T \to P(X)$ is *monotone nondecreasing* if $t \leqslant t'$ implies $M(t) \leqslant_s M(t')$. In particular, if the sets Z and Y are singletons, then the strong set order $\leqslant_s$ coincides with the given order $\leqslant$ on the underlying choice set, so that $\{z\} \leqslant_s \{y\}$ if and only if $z \leqslant y$. Also, if $M(t) = [0, g(t)] = \{x \in X\colon 0 \leqslant x \leqslant g(t)\}$, where g: $T \to X$ is increasing, then M is nondecreasing in the strong set order.

Note that the strong set order is not generally reflexive: $S \geqslant_s S$ if and only if S is a sublattice of X. This fact gives sublattices a particularly important role in our theory. The order is transitive on nonempty sets, but the empty set is an exception: for any set S, $S \leqslant_s \varnothing \leqslant_s S$.

Topkis (1976) develops a simple characterization of the structure of the sublattices of $\mathbb{R}^n$ in terms of a set of n^2 constraints, each of which involves no more than two components of the vector.

THEOREM 1: *A subset S of $\mathbb{R}^n$ is a sublattice if and only if there exist $n(n-1)$ functions g_{ij}: $\mathbb{R}^2 \to \mathbb{R}$ $(i \neq j)$, each of which is increasing in its first argument and decreasing in its second, and n sets $S_i \subset \mathbb{R}$ $(i = 1, \dots, n)$ such that $S = \{x | g_{ij}(x_i, x_j) \leqslant 0$ for all $1 \leqslant ij \leqslant n\} \cap \{x | x_i \in S_i\}$.*

The ranking of complete sublattices in the strong set order will prove especially useful in applications, and it can be given a simple characterization.

[3] There is also a topological characterization of completeness: A sublattice S of X is complete if and only if it is compact in the order-interval topology. On bounded sets in $\mathbb{R}^n$, the order-interval topology coincides with the Eucilidean topology (Birkoff (1967)).

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THEOREM 2: *S and S' are complete sublattices of X such that $S \geqslant_s S'$ if and only if there exists a complete sublattice R of X such that $S = R \cap \{x \geqslant \inf(S)\}$ and $S' = R \cap \{x \leqslant \sup(S')\}$. Moreover, if S and S' are complete sublattices with $S \geqslant_s S'$, then $S \cup S'$ is a complete sublattice.*

PROOF: Suppose S and S' are complete sublattices with $S \geqslant_s S'$. To establish that $S \cup S'$ is a sublattice, suppose $x, y \in S \cup S'$. If x and y are both elements of the same sublattice S or S', then their meet and join are elements of the same sublattice, and hence clearly elements of $S \cup S'$. If $x \in S$ and $y \in S'$, then by the definition of the strong set order, $x \wedge y \in S'$ and $x \vee y \in S$. Hence, $S \cup S'$ is a sublattice.

To establish that $S \cup S'$ is complete, let $Q \subset S \cup S'$. If either $Q \cap S$ or $Q \cap S'$ is empty, then the completeness of S and S' implies that $\inf(Q), \sup(Q) \in S \cup S'$. Now suppose both $Q \cap S$ and $Q \cap S'$ are nonempty and let $x_S = \inf(Q \cap S)$ and $x_{S'} = \inf(Q \cap S')$. Then $x_S \in S$ and $x_{S'} \in S'$, again by the completeness of S and S'. Then $\inf(Q) = x_S \wedge x_{S'} \in S' \subset S \cup S'$, because S is higher than S', and $\sup(Q) = x_S \vee x_{S'} \in S \cup S'$.

For the characterization of S, suppose $x \in S' \cap \{x \geqslant \inf(S)\}$. Then since $S \geqslant_s S'$ and $\inf(S) \in S$, we may conclude that $x = x \vee \inf(S) \in S$. Hence, $S = (S \cup S') \cap \{x \geqslant \inf(S)\}$. The characterization of S' is proved similarly.

The proof of the converse is immediate. *Q.E.D.*

According to this theorem, one complete sublattice is higher than another if and only if the first consists of the part of some sublattice lying above some point ($\inf(S)$) and the other consists of the part of the same sublattice lying below some other point ($\sup(S')$).

The preceding definitions have been standard ones. The critical new concept introduced here is the following one. Let X be a lattice, T be a partially ordered set, and f: $X \times T \to \mathbb{R}$. Then f satisfies the *single crossing property* in $(x; t)$ if for $x' > x''$ and $t' > t''$, $f(x', t'') > f(x'', t'')$ implies that $f(x', t') > f(x'', t')$ and $f(x', t'') \geqslant f(x'', t'')$ implies that $f(x', t') \geqslant f(x'', t')$. If $f(x', t'') \geqslant f(x'', t'')$ implies that $f(x', t') > f(x'', t')$ for every $t' > t''$, then f satisfies the *strict single crossing property* in $(x; t)$.

The terms "single crossing property" and "strict single crossing property" are used because the expression $f(x', t) - f(x, t)$, regarded as a function of t, crosses zero only once (and only from below) when these conditions hold. Notice that if we think of the set X as the choice space and the set T as the space of parameters, then not only is the single crossing property a condition describing the relationship between the choice variables and the parameters, but it is also an *ordinal* condition. This property is closely related to the single crossing condition used by Spence and Mirrlees in their signaling and optimal taxation models, and in many other models of modern incentive theory. In the now standard formulation, a decision maker of type t who chooses a point $(x, y) \in \mathbb{R}^2$ has a payoff of $U(x, y; t)$. A continuously differentiable function U on a rectangular domain with $U_y \neq 0$ satisfies the (strict) Spence-Mirrlees

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condition if $U_x/|U_y|$ is nondecreasing (increasing) in t for any fixed (x, y). We shall limit attention to *completely regular* functions U, that is, continuously differentiable functions for which the indifference sets $\{(x, y)|U(x, y, t) = \bar{u}\}$ are path-connected. This condition is always satisfied if U, in addition to being continuously differentiable, has both U_x and U_y nonzero everywhere.

The relationship between the Spence-Mirrlees single crossing property and our alternative definitions of the single crossing property is established by the following theorem.

THEOREM 3: *Let $\mathbb{R}^2$ be given the lexicographic order, with $(x, y) \geqslant (x', y')$ if either $x > x'$ or $x = x'$ and $y \geqslant y'$. Suppose that $U(x, y, t)$: $\mathbb{R}^3 \to \mathbb{R}$ is completely regular and twice continuously differentiable with $U_y \neq 0$. Then $U(x, y; t)$ has the (strict) single crossing property in $(x, y; t)$ if and only if it satisfies the (strict) Spence-Mirrlees condition.*

PROOF: We treat the strict case only; the other case is almost identical.

$\Leftarrow$: Suppose $(\bar{x}, \bar{y}) > (\hat{x}, \hat{y})$ in the lexicographic order and $U(\bar{x}, \bar{y}; t) \geqslant U(\hat{x}, \hat{y}; t)$. If $\bar{x} = \hat{x}$, then since U is strictly increasing or strictly decreasing in y, the sign of this inequality cannot depend on t, so we may limit our attention to the case where $\bar{x} > \hat{x}$. If, contrary to the theorem, there exists some $\bar{t} > t$ such that $U(\bar{x}, \bar{y}; \bar{t}) < U(\hat{x}, \hat{y}; \bar{t})$, then by continuity of U there exists some $\hat{t} < \bar{t}$ such that $U(\bar{x}, \bar{y}; \hat{t}) = U(\hat{x}, \hat{y}; \hat{t})$. Then, since U is completely regular, there is some isoutility curve $\{(x(s), y(s))|s \in [0, 1]\}$ such that $x'(s) > 0$, $(x(0), y(0)) = (\hat{x}, \hat{y})$, and $(x(1), y(1)) = (\bar{x}, \bar{y})$. But then the Spence-Mirrlees condition implies that

$$
\begin{aligned}
0 &= \frac{\left(\frac{d}{ds} U[x(s), y(s), \hat{t}]\right)}{|U_y[x(s), y(s), \hat{t}]|} \\
&= y'(s) \operatorname{sign}(U_y) + \frac{U_x}{|U_y|}[x(s), y(s), \hat{t}]\, x'(s) \\
&< y'(s) \operatorname{sign}(U_y) + \frac{U_x}{|U_y|}[x(s), y(s), \bar{t}]\, x'(s).
\end{aligned}
$$

So,

$$
\begin{aligned}
0 &< \int_0^1 \left(y'(s) \operatorname{sign}(U_y) + \frac{U_x}{|U_y|}[x(s), y(s), \bar{t}]\, x'(s) \right) \\
&\quad \times |U_y(x(s), y(s), \bar{t})|\, ds \\
&= \int_0^1 \frac{d}{ds} U(x(s), y(s), \bar{t})\, ds \\
&= U(\bar{x}, \bar{y}; \bar{t}) - U(\hat{x}, \hat{y}, \bar{t})
\end{aligned}
$$

contrary to the hypothesis.

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$\Rightarrow$: Consider a rectangular set $D \subset \mathbb{R}^2$ on which the Spence-Mirrlees condition does not hold, so that $U_x/|U_y|$ decreases (weakly) from t to $\bar{t}$. Let (x, y) and $(\bar{x}, \bar{y})$ be points in the interior of D such that $\bar{x} > x$ and $U(x, y, t) = U(\bar{x}, \bar{y}, t)$. Then repeating the same steps leads to the conclusion that $U(x, y, \bar{t}) \geqslant U(\bar{x}, \bar{y}, \bar{t})$, contrary to the strict single crossing condition. *Q.E.D.*

When the choice set is totally ordered (a *chain*), the single crossing property is the only condition we will need for comparative statics. However, when the choice set is not totally ordered, an additional condition is necessary. For example, when the choice set is $\mathbb{R}^n$, monotone comparative statics restricts the interactions among the components of the choice variable. Thus, given a lattice X, we say that a function f: $X \to \mathbb{R}$ is *quasisupermodular* if (i) $f(x) \geqslant f(x \wedge y)$ implies $f(x \vee y) \geqslant f(y)$ and (ii) $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$. Note that when $X = \mathbb{R}$, so that the choice set is a chain, every function is quasisupermodular, as the order operations meet and join are then trivial. When $X = \mathbb{R}^2$, requiring f to be quasisupermodular is equivalent to requiring that f satisfy the single crossing property in $(x_1; x_2)$ and also in $(x_2; x_1)$. When the choice space is of greater dimension, quasisupermodularity involves additional multivariate restrictions as well. Quasisupermodularity expresses a weak kind of complementarity between the choice variables; if an increase in some subset of the choice variables is desirable at some level of the remaining choice variables, it will remain desirable as the remaining variables also increase. With the definitions in place, we can state our main theorem.

THEOREM 4 (Monotonicity Theorem): *Let f: $X \times T \to \mathbb{R}$, where X is a lattice, T is a partially ordered set and $S \subset X$. Then* $\arg\max_{x \in S} f(x, t)$ *is monotone nondecreasing in (t, S) if and only if f is quasisupermodular in x and satisfies the single crossing property in $(x; t)$.*

PROOF: Let $M(t, S) \equiv \arg\max_{x \in S} f(x, t)$.

$\Leftarrow$: Let $S' \geqslant_s S$, $t \leqslant t'$, $x \in M(t, S)$, $x' \in M(t', S')$. Consider $x \vee x'$. Since $x \in M(t, S)$ and $S \leqslant_s S'$, then $f(x, t) \geqslant f(x \wedge x', t)$. By quasisupermodularity, $f(x \vee x', t) \geqslant f(x', t)$, and by the single crossing property, $f(x \vee x', t') \geqslant f(x', t')$, hence $x \vee x' \in M(t', S')$.

Similarly, consider $x \wedge x'$. Since $x' \in M(t', S')$ and $S \leqslant_s S'$, then $f(x', t') \geqslant f(x \vee x', t')$, or $f(x \vee x', t') - f(x', t') \leqslant 0$. Then the single crossing property implies that $f(x \vee x', t) - f(x', t) \leqslant 0$, so quasisupermodularity implies $f(x \wedge x', t) \geqslant f(x, t)$, i.e., $x \wedge x' \in M(t, S)$.

$\Rightarrow$: To show that f is quasisupermodular, suppose t is fixed and x, x' are in X. Let $S \equiv \{x, x \wedge x'\}$, and let $S' \equiv \{x', x \vee x'\}$. Then $S \leqslant_s S'$. If $f(x, t) \geqslant (>) f(x \wedge x', t)$, then $x \in M(t, S)$, hence $f(x \vee x', t) \geqslant (>) f(x', t)$.

To show that the single crossing property must hold, let $S \equiv \{x, \bar{x}\}$, where $\bar{x} \geqslant x$. $f(\bar{x}, t) - f(x, t) \geqslant (>) 0 \Rightarrow \bar{x} \in M(t, S) \leqslant_s M(\bar{t}, S)$ for $\bar{t} \geqslant t$, so $f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geqslant (>) 0$ for every $\bar{t} \geqslant t$. *Q.E.D.*

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COROLLARY 1: *$f(x)$ is quasisupermodular if and only if* $\arg\max_{x \in S} f(x)$ *is monotone nondecreasing in S.*

PROOF: Apply Theorem 4, setting $T \equiv \{t\}$. *Q.E.D.*

Thus the restrictions on how f varies in the choice variable are implied by monotonicity of the set of maximizers in the constraint set. If the conclusion that the set of maximizers is monotone nondecreasing in S were not required, then f need not be quasisupermodular. This observation motivates many of the partial monotonicity results developed in the next section.

The next corollary describes the structure of the set of optimizers.

COROLLARY 2: *If S is a sublattice of X, and f is quasisupermodular, then* $\arg\max_{x \in S} f(x, t)$ *is a sublattice of S.*

PROOF: This corollary also follows directly from the theorem, as $(t, S) \leqslant (t, S)$ if and only if S is a sublattice of X. *Q.E.D.*

Of course we have made no claim yet that the set of maximizers of f is nonempty; without such results we could certainly just be studying properties of the empty set. For many applications, the existence of maximizers will follow immediately from the upper semi-continuity of f and compactness of S. When, in addition, f is quasisupermodular and S is a sublattice, the set of maximizers is actually a complete sublattice, and hence has a greatest and least element $x^*(t, S)$ and $x_*(t, S)$. The monotonicity theorem then guarantees that $x^*(t, S)$ and $x_*(t, S)$ are nondecreasing functions, and hence monotone selections from the set of maximizers. General sufficient conditions for the set of maximizers to be a nonempty complete sublattice are given in the appendix.

A variant of Theorem 4 corresponding to the strict single crossing property can be proved similarly.

THEOREM 4′ (Monotone Selection Theorem): *Let f: $X \times T \to \mathbb{R}$, where X is a lattice and T is a partially ordered set. If S: $T \to 2^X$ is nondecreasing and if f is quasisupermodular in x and satisfies the strict single crossing property in $(x; t)$, then every selection $x^*(t)$ from* $\arg\max_{x \in S(t)} f(x, t)$ *is monotone nondecreasing in t.*

3. CHARACTERIZING QUASISUPERMODULARITY AND THE SINGLE CROSSING CONDITION

Although quasisupermodularity and the single crossing property may seem abstract and difficult to check, the results of this section give several important characterizations of these conditions in terms of previously known conditions: supermodularity, increasing differences, and the Spence-Mirrlees single crossing condition.

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Given a lattice X and a partially ordered set T, the function f: $X \to \mathbb{R}$ is *supermodular* if for all $x, y \in X$, $f(x)+f(y) \leqslant f(x \vee y)+f(x \wedge y)$, and the function f: $X \times T \to \mathbb{R}$ has *increasing differences* in (x, t) if for $x' \geqslant x$, $f(x', t) - f(x, t)$ is monotone nondecreasing in t. In Euclidean applications, supermodularity means that increasing any subset of the decision variables raises the incremental returns associated with increases in the others. Similarly, increasing differences means that increasing a parameter raises the marginal return to activities. An important comparative statics theorem, due to Topkis (1978), is the following one.

THEOREM 5: *Let X be a lattice, T a partially ordered set, and f: $X \times T \to \mathbb{R}$. If $f(x, t)$ is supermodular in x and has increasing differences in $(x; t)$, then* $\arg\max_{x \in S} f(x, t)$ *is monotone nondecreasing in (t, S).*

It should be clear, either from the definitions or from Theorems 4 and 5, that any supermodular function is also quasisupermodular, and similarly, any function which has increasing differences in (x, t) will also satisfy the single crossing property in (x, t) as well.

Supermodularity and increasing differences are often especially useful in applications because they have a number of strong properties. As the following theorem of Topkis (1978) indicates, supermodularity and increasing differences are easily characterized for smooth functions on $\mathbb{R}^n$.

THEOREM 6: *Let f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be twice continuously differentiable on the interval (a, b). Then f has increasing differences in (x, t) if and only if $\partial^2 f / \partial x_i \partial t_j \geqslant 0$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$; and f is supermodular in x if and only if $\partial^2 f / \partial x_i \partial x_j \geqslant 0$ for $i \neq j$.*

Furthermore, as the next theorem illustrates, supermodularity is preserved under a number of operations. This theorem is based on results in Topkis (1978).

THEOREM 7: (i) *If f and g are supermodular, and $\alpha, \beta \geqslant 0$, then $\alpha f + \beta g$ is supermodular.* (ii) *If the functions $f_1, f_2, \ldots$ are supermodular and f^* is the pointwise limit of $\{f_n\}$, then f^* is supermodular. Also, $\sum_{n=1}^{\infty} f_n$ is supermodular if this pointwise sum is well defined.* (*iii*) *If f is supermodular and increasing and g: $\mathbb{R} \to \mathbb{R}$ is increasing and convex, then $g \circ f$ is supermodular.*

Notice that Theorem 7 indicates that there may be a role for convex functions in lattice theoretic comparative statics analyses, even though both the preferred sets and the constraint sets need not be convex.

Clearly, for any quasisupermodular function f: $X \to \mathbb{R}$ and any strictly increasing function g: $\mathbb{R} \to \mathbb{R}$, the composition $g(f(x))$ is also quasisupermodular. In particular, if f is supermodular, then $g(f(x))$ is quasisupermodular for any strictly increasing function g, and if there exists some such strictly increasing

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x	y			
	0	1	2	3
0	1	2	2	1
1	3	4	5	3

EXAMPLE 1—The tabulated function f is quasisupermodular but not supermodularizable.

function h: $\mathbb{R} \to \mathbb{R}$ such that $h(f(x))$ is supermodular, then the original function f is quasisupermodular. Such functions are called *supermodularizable*, and have been studied by Li Calzi (1991). Supermodularizable functions can be usefully applied in one of the monopoly pricing problems discussed in the following section.

Although supermodular and supermodularizable functions are an important subset of quasisupermodular functions, not every quasisupermodular function is supermodularizable, as Example 1 illustrates. To see this, suppose g is an increasing function such that $g(f(\cdot))$ is supermodular. Then, focusing on the domain $\{0,1\} \times \{0,1\}$, supermodularity implies that $g(3)+g(2) \leqslant g(4)+g(1)$. Focusing on $\{0,1\} \times \{2,3\}$, we have $g(5)+g(1) \leqslant g(3)+g(2)$. Together these imply that $g(5) \leqslant g(4)$, contrary to the hypothesis that g is increasing. The same example establishes that functions with the single crossing property need not be supermodularizable. The actual relation between supermodularity and quasisupermodularity involves transformations on restricted domains.

LEMMA: *Suppose f: $X \to \mathbb{R}$ is quasisupermodular, where $X = \{x, x', x \vee x', x \wedge x'\}$. Then there exists some h: $\mathbb{R} \to \mathbb{R}$ such that h is strictly increasing and $h \circ f$: $X \to \mathbb{R}$ is supermodular.*

PROOF: Let f be the function which maps $x \to a$, $x \vee x' \to b$, $x \wedge x' \to c$, and $x' \to d$. Without loss of generality, assume that $d \geqslant c$. Quasisupermodularity implies that $b \geqslant a$ and either $a \leqslant c$ or $b \geqslant d$. In the first case, $a = \min(a,b,c,d)$ and in the second, $b = \max(a,b,c,d)$.

Suppose $a = \min(a,b,c,d)$. If $a < b,c,d$, set h to be the identity of $\{b,c,d\}$ and $h(a) = e$, where $e < \min(b+c-d, a)$. Otherwise (if a is not the unique minimum value), let h be the identity function as $a = b \Rightarrow c \geqslant d$, $a = c \Rightarrow b \geqslant d$, $a = d \geqslant c \Rightarrow a = c \Rightarrow b \geqslant d$. Then, $h \circ f$ is supermodular on X. A similar argument applies when $b = \max(a,b,c,d)$. *Q.E.D.*

The above lemma provides the heart of a characterization of quasisupermodular functions, suggesting a necessary and sufficient condition.

THEOREM 8: *Let X be a lattice and f: $X \to \mathbb{R}$. Then f is quasisupermodular if and only if there exists some g: $\mathbb{R} \times X \times X \to \mathbb{R}$ such that* (i) *$g(r, x_1, x_2)$ is strictly increasing in r for every fixed $(x_1, x_2) \in X^2$ and* (ii) *for every $x_1, x_2 \in X$, $g(f(x), x_1, x_2)$ is supermodular in x on the sublattice $\{x_1, x_2, x_1 \vee x_2, x_1 \wedge x_2\}$.*

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PROOF: One direction follows from the preceding lemma. For the other, suppose such a transformation g exists and that $x, x' \in X$ are such that $f(x) \geqslant (>) f(x \vee x')$. Then $g(f(x), x, x') \geqslant (>) g(f(x \vee x'), x, x')$, and g supermodular on $\{x, x', x \vee x', x \wedge x'\} \Rightarrow g(f(x \wedge x'), x, x') \geqslant (>) g(f(x'), x, x')\}$. Then since g is strictly increasing on this lattice, $f(x \wedge x') \geqslant (>) f(x')$. *Q.E.D.*

Similarly, one can establish the following theorem.

THEOREM 9: *Let X be a lattice and T a partially ordered, finite set and f: $X \times T \to \mathbb{R}$. Then f has the single crossing property if and only if there exists g: $\mathbb{R} \times X^2 \times T \to \mathbb{R}$ such that g is increasing in its first argument and for all $x_1 \geqslant x_2$ in X, $g(f(x, t), x_1, x_2, t)$ has increasing differences in $(x; t)$ on the set $\{x_1, x_2\} \times T$.*

As a particular case of Theorems 8 and 9, given a function f: $X \times T \to \mathbb{R}$, if there exists a function g: $\mathbb{R} \times T \to \mathbb{R}$ such that g is increasing in its first argument for every t and such that $g(f(x, t), t)$ is supermodular in x and has increasing differences in (x, t), then f is quasisupermodular in x and satisfies the single crossing property in (x, t). When this condition holds, we call the function f an *extended supermodularizable* function. A monopoly pricing problem in the next section illustrates the application of such functions.

Although supermodularity of the objective function is never necessary for monotone comparative statics, in some problems characterized by separable objective functions, supermodularity of one term may sometimes be necessary in order for the objective function to be quasisupermodular or satisfy the single crossing property.

THEOREM 10: *Let f: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Then $f(x, t) + p \cdot x$ is quasisupermodular in x and has the single crossing property in $(x; t)$ for all $p \in \mathbb{R}^n$ if and only if f is supermodular. If $f(\cdot)$ is nondecreasing in x, then $f(x, t) - w \cdot x$ is quasisupermodular in x and has the single crossing property in $(x; t)$ for all nonnegative $w \in \mathbb{R}^n$ if and only if f is supermodular.*

PROOF: If f is supermodular in (x, t) then $g(x, t, p) \equiv f(x, t) + p \cdot x$ is supermodular and therefore quasisupermodular in (x, t, p), so it also has the less restrictive properties described in the theorem.

If f is not supermodular in x, then (suppressing the argument t) there exist some x and y so that $f(x) + f(y) > f(x \vee y) + f(x \wedge y)$. Choose p so that $p \cdot [x - (x \wedge y)] + f(x) - f(x \wedge y) = 0$. By inspection, for this choice of p, $f + p \cdot x$ is not quasisupermodular.

Similar arguments cover the other cases. *Q.E.D.*

Dissection

In problems with a single real-valued choice variable, the objective function is always quasisupermodular in that variable, and one only needs to verify that the

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single crossing property holds to apply our theory. The most flexible method for verifying that the single crossing property holds in such problems is the *method of dissection*. This method builds on the relationship between the single crossing property and the Spence-Mirrlees single crossing condition established in the previous section by separating the effects of changes in the choice variables into two effects and embedding the problem in a family of problems in which one of the effects remains unchanged. We do this by introducing the idea of a richly parameterized family of functions, which is a family that contains at least one member passing through any pair of points. Formally, $\{h(\cdot;\alpha): \mathbb{R} \to \mathbb{R}\}$ is *richly parameterized* if for all (x', y') and (x'', y'') with $x' \neq x''$, there is some $\hat{\alpha}$ such that $y' = h(x';\hat{\alpha})$ and $y'' = h(x'';\hat{\alpha})$. For example, given any function $h_0: \mathbb{R} \to \mathbb{R}$, the family $\{h_0(x) + \alpha_1 x + \alpha_0 | \alpha \in \mathbb{R}^2\}$ is richly parameterized.

THEOREM 11: *Let $U(x, y, t): \mathbb{R}^3 \to \mathbb{R}$ be completely regular with $U_y \neq 0$ and let $h(\cdot;\alpha): \mathbb{R} \to \mathbb{R}$ be a richly parameterized family. Then U satisfies the* (*strict*) *Spence-Mirrlees condition if and only if for all α, $g(x;t,\alpha) = U(x, h(x;\alpha), t)$ has the* (*strict*) *single crossing property in $(x;t)$.*

PROOF: Observe that whenever the single crossing property holds on a lattice, it also holds on all of its sublattices. Also, the function $g(x;t,\alpha)$ is the restriction of U to the sublattice $\{(x,y)|y = h(x;\alpha)\}$ with the lexicographic order. According to Theorem 3, if U is completely regular and satisfies the Spence-Mirrlees single crossing condition, it has the single crossing property, so g has the single crossing property in $(x;t)$ as well.

If the Spence-Mirrlees condition fails, then there exists an open rectangular neighborhood $D \subset \mathbb{R}^3$ in which $U_x/|U_y|$ is decreasing in t. By Theorem 3, one may choose points (x', y', t') and (x'', y'', t') in D satisfying $x' > x''$, $t'' > t'$, $U(x', y', t') > U(x'', y'', t')$ and $U(x', y', t'') < U(x'', y'', t'')$. Choose α so that $y' = h(x';\alpha)$ and $y'' = h(x'';\alpha)$. Then $g(x';t',\alpha) > g(x'';t',\alpha)$ but $g(x';t'',\alpha) < g(x'';t'',\alpha)$, contradicting the single crossing property.

The proofs for the strict cases are similar. *Q.E.D.*

Theorem 11 studies how changing trade-offs affect comparative statics. Here the objective function can be expressed as depending on the variable x in two ways, one of which is of known sign. Suppose that known effect is negative. Optimization involves trading off that cost against the net benefit, if any, associated with the other effects of x. An intuitive principle is that a parameter change that increases the relative significance of any beneficial effects will lead to a higher optimal choice of x; the direct conclusion of Theorem 11 gives a formal statement of that principle. In the reverse direction, the theorem asserts that if there is a sufficiently rich parameterization of the costly effect, then no weaker condition guarantees monotone comparative statics for the whole parameterized class.

To illustrate the application of Theorem 11, consider the effects of a short run increase in the market size on the monopoly price. Let the number of

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customers be N. The firm's problem is to choose the quantity q to sell per customer to maximize $\pi(q; N) \equiv NqP(q) - C(Nq)$, subject to $q \in K$. Without some restrictions on demand, one cannot ensure that this function is either supermodular in $(q, -N)$ or concave in q. Nevertheless, the comparative statics analysis is simple: $q^*(N)$ is nonincreasing for all choices of K and P if the cost function C is convex, and nondecreasing if the cost function is concave. The properties of the demand function are irrelevant. On the necessity side, we may embed the firm's problem in a family of problems with inverse demand $P(q) + p$ and costs $C(q) + cq$, where $p, c \geqslant 0$, by taking $h(q; \alpha_1, \alpha_2) = qP(q) + \alpha_1 q + \alpha_2$ where $\alpha_1 = p - c$, α_2 is arbitrary, and $U(x, y; t) = ty - C(tx)$. Then $\pi(q; N, \alpha_1, \alpha_2) = U(q, h(q; \alpha_1, \alpha_2); N)$. According to Theorem 11 and Theorem 4, $q^*(N; \alpha_1, \alpha_2)$ is nondecreasing in N for all values for α_1, α_2, and K if and only if C is concave, and nonincreasing for all such values if and only if C is convex. No restrictions at all are imposed on the demand function P in reaching this conclusion.

For a more elaborate application, consider the Bertrand model with differentiated products. Suppose there are N firms, indexed by n. The profit function for firm n is given by

$$\pi_n(p_n, p_{-n}) = p_n D_n[p_n, p_{-n}] - C_n(D_n[p_n, p_{-n}]),$$

where p_n is firm n's price, p_{-n} is the vector of competitors' prices, C_n is a continuous, increasing function, D_n is continuously differentiable and decreasing in p_n and firm n's demand becomes increasingly inelastic with increases in p_{-n}, that is, $\log(D_n(p_n; p_{-n}))$ has increasing differences. The game with this specification for each firm was studied by Milgrom and Roberts (1990b) for the case of constant marginal costs. They showed that the objective function is then supermodularizable, which implies that it satisfies the single crossing property in $(p_n; p_{-n})$. We extend this last conclusion to the case of nonlinear cost functions.

To check the single crossing property directly for this case, fix two vectors of prices p' and p'' with $p'_n > p''_n$ and $p'_{-n} > p''_{-n}$ and for $p_{-n} \in \{p'_{-n}, p''_{-n}\}$ define $c(p_{-n})$ and $K(p_{-n})$ to equate $C_n(q)$ to $qc(p_{-n}) + K(p_{-N})$ for $q \in \{D_n(p'_n, p_{-n}), D_n(p''_n, p_{-n})\}$. On the relevant domain, the profit function satisfies $\pi_n(p_n, p_{-n}) = (p_n - c(p_{-n}))D_n(p_n, p_{-n}) - K(p_{-n})$.

There are two cases, depending on whether $p''_n \leqslant c(p'_{-n})$ or $p''_n > c(p'_{-n})$. In the first case, since $p'_n > p''_n$, $\pi_n(p'_n, p'_{-n}) > \pi_n(p''_n, p'_{-n})$. In that event, the single crossing condition is satisfied for these prices. For the second case, let $U(x, y, p_{-n}) = (y - c(p_{-n}))D_n(x, p_{-n}) - K(p_{-n})$ and $h(x) = x$.[4] Restrict y so that $y \geqslant c(p'_{-n})$. Then U is completely regular because it is nondecreasing in x. A routine calculation shows that U satisfies the Spence-Mirrlees condition on this domain if and only if $c(\cdot)$ is nondecreasing, which occurs if the goods are

[4] We may treat p_{-n} as a scalar variable since its values in this analysis form a chain.

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substitutes and C is convex or if the goods are complements and C is concave.[5] With either combination of assumptions, the single crossing property is satisfied for any specified prices.

We shall study the implications of this property for equilibrium in Bertrand models in Section 5.

4. METHODS FOR APPLICATIONS

The method of dissection just discussed is one of three main techniques for transforming problems into forms where our theorems apply. Two other important techniques, aggregation and selective ordering, and two secondary techniques, parameter contingent transformations and composite functions, are described in this section.

Aggregation

The most important method is based on the idea of aggregation, which has been used repeatedly in the past by other researchers. Hicks used aggregation in the form of composite commodities to simplify his treatment of consumer and producer theory; Koopmans used it when he introduced recursive utility functions that allowed future consumption to be considered as an aggregate against which to trade off current consumption; and Topkis (1978) used it in his dynamic programming formulation of network flow problems to trade off flows on one arc against the aggregate of the substitute flows. Indeed, as we use the term, "aggregation" is synonymous with dynamic programming.

For our purposes, the basic principles are expressed by the following two results.[6]

COROLLARY 3 (Aggregation Principle): *Let X be a lattice, T a partially ordered set, Y an arbitrary set with $W \subset Y$, f: $X \times Y \times T \to \mathbb{R}$, and $x^*(t,S)$ be* $\arg\max_{x \in S} \max_{y \in W} f(x, y; t)$. *Then $x^*(t,S)$ is monotone nondecreasing in (t,S) if and only if $g(x;t) \equiv \max_{y \in W} f(x,y;t)$ is quasisupermodular in x and satisfies the single crossing property in (x,t).*

PROOF: This follows immediately from Theorem 4. *Q.E.D.*

COROLLARY 4: *Let f: $\mathbb{R}^2 \to \mathbb{R}$ and h: $\mathbb{R} \times Y \to \mathbb{R}$ and define $x^*(t,S,p) =$* $\arg\max_{x \in S} \max_{y \in W} f(x;t) + h(x,y) + px$. *Then $x^*(t,S,p)$ is nondecreasing in t for all $S \subset \mathbb{R}$ and all $p \in \mathbb{R}$ if and only if f is supermodular. If $f(x;t) + h(x,y)$ is*

[5] Intuitively, $c(p_{-n})$ is the average incremental cost over an interval of outputs that depends on p_{-n}. If the goods are substitutes, then the limits of the range are increasing and so the average incremental costs increases if C is convex. If the goods are complements, then the limits are decreasing and cost increases if C is concave.

[6] See also the related "Reduced Forms Theorem" of Milgrom, Qian, and Roberts (1991).

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nondecreasing in x for all y, then $x^*(t, S, p)$ is nondecreasing in t for all $S \subset \mathbb{R}$ and all $p \leqslant 0$ if and only if f is supermodular.

PROOF: Since every function of one real variable is quasisupermodular, $x^*(t, S, p)$ is nondecreasing in S without any conditions on f. Consequently, being nondecreasing in (t, S) is logically equivalent to being nondecreasing in t for all S. Now apply Corollary 3 and Theorem 10. *Q.E.D.*

Corollary 3 makes it possible to focus attention on the variable x when monotone comparative static conclusions about y are unavailable. Equally important, it eliminates the constraint set W from the problem, which is necessary to study situations where $x^*(t, S, W)$ is monotone in (t, S) but not in W. The historical roots and theoretical significance of the aggregation principle are discussed in more detail in the second example below.

The direct implication of Corollary 4 formalizes the intuitive principle that increasing the marginal return to a single variable x in an optimization problem leads to a higher optimal value of that variable. The reverse implication concerns a family of models in which the objective is the sum of two terms, one parameterized by t and the other richly parameterized by p, so that the marginal returns to x in the second term vary over a wide range. In that case, unless increases in the parameter increase marginal returns over all ranges of the variable, it is not possible to draw the same conclusion about monotone comparative statics without imposing specific and restrictive assumptions about the parameter p.

Aggregation plays a central role in analytical methods based on lattice theory. Indeed, according to Theorem 1, the sublattices of $\mathbb{R}^n$ include only sets that can be described by constraints involving at most two choice variables at a time. In a number of economic problems such as consumer decision problems with three or more goods, consumption-savings problems over an infinite horizon, or network flow problems requiring the total flow from three or more sources to a final node to satisfy that node's requirements, the usual formulation of the problem involves constraints on more than two variables simultaneously. To apply the lattice methods, one approach is to reformulate the problem, isolating a variable of interest such as consumption or investment or a flow variable and representing the constraint as one involving just that variable and an aggregate of the others. We shall illustrate this approach in the context of the Ramsey-Cass-Koopmans consumption-investment model.[7]

Consider the problem of maximizing the utility of consumption over an infinite horizon starting with initial resources k_0 and assuming that, in any period, s units of savings can be converted into $f(s)$ units of resources at the start of the next period. A final constraint is that $0 \leqslant s \leqslant k$. Koopmans gave the most general tractable formulation of this problem, assuming that utility for future consumption was stationary over time, independent of past consumption

[7] See Cass (1965) and Koopmans (1960, 1965).

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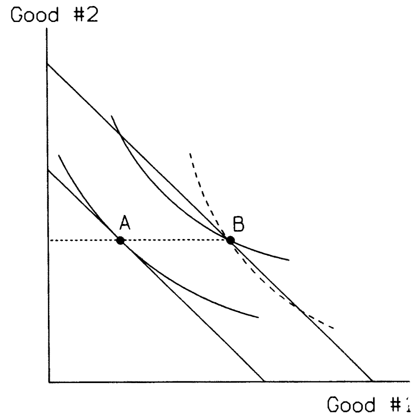

FIGURE 1.—Good #2 is normal when the indifference curve is flatter at B than at A.

levels, and involved some "impatience," and concluding that it could be written in the form: $U(c_0, c_1, \dots) = W(c_0, u)$ where $u = U(c_1, c_2, \dots)$. Notice that u is an aggregate which substitutes in the problem for the details of future consumption. We assume that $W_1, W_2 > 0$.

One important question in this theory is: When will current savings be a nondecreasing function of current levels of capital? The most general known answers have been obtained using revealed preference theory,[8] but a straightforward application of the ordinal theory yields an alternative answer: regardless of the technology f, savings will be a nondecreasing function of current capital if, in the consumer's aggregate preference function W, the future consumption aggregate u behaves like a normal good. Figure 1 illustrates the normal goods condition for the standard convex consumption model with linear budget sets. Given the utility function W, good #2 will be a normal good for all prices and income levels if and only if W_1/W_2 is a decreasing function of consumption of the first good, where subscript i denotes a partial derivative with respect to argument i. As we show below, the same concept can be applied without convexity restrictions to this consumption-savings model to yield the desired comparative statics conclusions.

Given current resources k and savings s, current consumption is $k - s$ and the maximum utility of future consumption is some amount $u(s)$, determined jointly by the technology and the consumer's utility function. The consumer's problem is one of maximizing $W(k - s, u(s))$. Applying the method of dissection (Theorem 11), with $U(x, y, t) = W(t - x, y)$ and $h(\cdot) = u(\cdot)$, we find that a sufficient condition for $s^*(k)$ to be nondecreasing is that $W_1(c, u)/W_2(c, u)$ be decreasing in c, that is, that the utility of future consumption enters W like a normal good. The method is also helpful for identifying conditions under which this normal goods assumption is necessary, but we do not pursue that here.

[8] See Benhabib, Majumdar, and Nishimura (1985).

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Selective Ordering

As we mentioned several times above, the lattice theoretic approach to monotone comparative statics relies critically on the notions of order imposed on the choice variables and parameters. The particular notions of order used determine the meanings of the single crossing property and quasisupermodularity and also the meaning of the monotone comparative statics conclusion. By appropriately defining an order on the choice variables and parameters, one can make use of the full flexibility of this theory.

For example, consider the standard neoclassical model of the firm with two inputs, capital and labor. The firm solves

$$\text{Maximize } pf(k,l) - wl - rk \quad \text{subject to} \quad k \in K \quad \text{and} \quad l \in L$$

where the sets K and L represent any constraints on the availability of capital or labor. The sets K and L need not be convex.

It is well known that increasing w reduces l. In the ordinal theory, this follows from Corollary 4 and Theorem 4 where the parameter is $-w$ rather than w. The mixed partial derivative of the objective function with respect to l and $-w$ is positive, so l is a nondecreasing function of $-w$, that is, a nonincreasing function of $-w$. The other direct price effects (r on k and p on output) follow similarly from Theorem 4 and Corollary 4.

To consider whether the inputs are substitutes or complements, we focus on the mixed partial derivative f_{kl}. If $f_{kl} \geqslant 0$, then the objective function is supermodular in $(k, l, -w, -r)$, so increases in r lead to reductions in both k and l. According to Corollary 4, there is no weaker sufficient condition to ensure, for example, that $k^*(w)$ is nondecreasing without more information about K or r. If $f_{kl} \leqslant 0$, then the objective function is supermodular in $(k, -l, w, -r)$ so, for example, an increase in w leads to an increase in k. Once again, no convexity is used[9] and the most general possible result is obtained.

The last standard question in the theory concerns the output expansion path, for which the standard intuitive analysis uses production isoquants. Thus, define $L(k,x)$ to be the minimum amount of labor required to produce (at least) x units of output using k units of capital. The objective is then $px - rk - wL(k,x)$. By Theorem 4 and Corollary 4, $k^*(x)$ is nondecreasing (capital is a *normal input*) if $L_{kx} \leqslant 0$ and $k^*(-x, K)$ is nondecreasing if $L_{kx} \geqslant 0$. These conclusions hold even if capital is indivisible—a basic fact about demand theory which appears to have been previously unknown. Corollary 4 goes further, asserting that there is no weaker sufficient condition to imply these conclusions if the cost of capital r and the possible lumps of capital K are to be left unspecified. The interpretation of the conditions is straightforward: L_k is the slope of the

[9] Note that the standard approach of differentiating dual profit functions depends on those functions being differentiable. Convexity of the underlying technology is a necessary but not sufficient condition for such differentiability. The unelaborated dual profit function approach is therefore less general than either the ordinal approach or the revealed preference approach.

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isoquant for output level x at capital level k, so the normal inputs condition is just the usual one about marginal rates of transformation.

Selective ordering does not refer only to changing signs of variables. It may also correspond, for example, to a propitious change of basis in a linear space, such as using capital levels rather than investment flows as the choice variables in a multiperiod capital investment problem. See the Arm's Race Game in Milgrom and Roberts (1990b) for an example of that sort.

Parameter Contingent Transformations

According to Theorems 8 and 9, quasisupermodularity and the single crossing property hold exactly for those functions for which there exists a parameter contingent transformation that makes the functions supermodular or have increasing differences on certain limited sets. Frequently, it is easy to find parameter-contingent transformations that apply over the whole domain of the function. The two that have arisen most often in our experience are the *log transformation*, which converts a multiplicatively separable problem into an additively separable one to which Corollary 4 applies, and the *scale transformation*. We illustrate both here.

For example, consider the problem of a monopolist who sets a price p facing demand $D(p)$ and with constant marginal cost c. The firm's profit is $(p-c)D(p)$. This is multiplicatively separable, and its logarithm is $\log(p-c)+\log(D(p))$. This is (strictly) supermodular in (p,c) regardless of the demand function D, so every selection from the optimal pricing function $p^*(c)$ is nondecreasing, and the same is true even if prices are constrained to be chosen in discrete monetary units. The intuition is that an increase in c increases the marginal rate of return to p (rather than the return itself) and so favors an increase in price.

To illustrate the scale transformation, we return to the example of a short run increase in the market size on the monopoly price which we treated previously using the method of dissection. The family of objective functions studied was $\pi = Nq(P(q)+p) - C(Nq) - Ncq$. We divide this objective by N to obtain the objective function $q(P(q)+p) - C(Nq)/N - cq$. This is to be maximized by choosing $q \in K$. By Corollary 4, $q^*(N)$ is nondecreasing for all values of p, c, and K if and only if $-C(Nq)/N$ is supermodular, that is, if and only if C is concave. Similarly, $q^*(N)$ is nonincreasing for all values of p, c, and K if and only if C is convex.

In smooth problems with a single real choice variable, these transformation methods apply only if the method of dissection also applies,[10] so in one sense the method of dissection subsumes these methods. However, the transformation methods often reflect the intuition of the analysis more clearly than the method of dissection. In the scale transformation application, for example, an obvious

[10] In the monopoly pricing problem discussed using the log transformation, the method of dissection can be applied by taking $U(x,y,t) = (x-c)y$ and $h(x) = D(x)$.

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intuitive analysis is that when costs are convex, increasing the market size is very much like increasing the marginal cost of units for an individual customer, and higher marginal costs naturally lead to lower quantities and higher prices if $P(\cdot)$ is downward sloping. The first step of the intuitive logic corresponds to the formal step of dividing the objective by N in order to focus on the *per customer* objective function; the second to noticing that increases in N raise marginal cost, that is, verifying that the transformed objective is supermodular in $(q, -N)$. In both the intuitive and ordinal logic, the next step is to draw the comparative statics conclusion. This formal agreement closely mirrors the economic intuition.

Composite Functions

The last two examples have been one-variable choice problems. Even when the multivariate theory does not apply directly, multivariable choice problems can often be reduced to a sequence of single variable choice problems. The problem is then reduced to one of composing monotone functions. The solution of the Ramsey-Cass-Koopmans problem can be interpreted in this way.

As another example, consider the short-run problem of a monopolist who finds a new and less expensive way of expanding its market, perhaps by adding new outlets or by some more effective advertising or promotion. The market size N, a parameter in the previous monopoly pricing example, is a choice variable in this problem. The firm's problem is to choose (q, N) to maximize $NqP(q) - C(Nq) - K(N, t)$ where K is submodular $(K_{Nt} \leqslant 0)$. By Corollary 4, $N^*(t)$ is nondecreasing, and by the previous analysis, $q^*(N, t) = q^*(N)$ is nonincreasing provided that C is convex. If, in addition, the demand function P is decreasing, then the firm's response to the change is to raise its price.

5. EQUILIBRIUM THEORIES

So far, we have limited attention to comparative statics for optimization problems. However, the ordinal approach is also useful for equilibrium problems, and in particular, to games with strategic complementarities, studied by Bulow, Geanakoplos, and Klemperer (1985), Milgrom and Roberts (1990b), Topkis (1979), and Vives (1990). These papers develop the notion of supermodular games, a class of games in which the players' strategy sets S_n are compact sublattices and the payoff functions $\pi_n(x_n, x_{-n}, t)$ are upper semicontinuous in the player's own strategy x_n, continuous in the competitor's strategies x_{-n}, and supermodular in $(x_n, x_m)(m \neq n)$ and (x_n, t) for any fixed values of the other variables. A large number of the most studied noncooperative games in economics have this structure, as demonstrated by the wide range of applications found in the papers cited above. In addition, this class of games has a series of useful properties, including existence of pure strategy equilibria, monotone comparative statics on equilibrium sets, coincidence of the predictions of various solution concepts (including Nash equilibrium, correlated equilibrium,

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and rationalizable strategies), stability under adaptive learning, and certain welfare properties, to name the main ones. As we will see, these properties can all be generalized to a larger class of games.

Consider the following general environment in which a certain class of games will be defined. A nonempty set N indexes the players, and each player's strategy set is S_n, partially ordered by $\geqslant_n$. The space of strategy profiles is then S, and player n has payoff function $\pi_n(x_n, x_{-n})$. Such a game has (ordinal) *strategic complementarities* if for every n:

(1) S_n is a compact lattice;

(2) π_n is upper semi-continuous in x_n for x_{-n} fixed, and continuous in x_{-n} for fixed x_n;

(3) π_n is quasisupermodular in x_n and satisfies the single crossing property in $(x_n; x_{-n})$.

Certainly the results of the previous sections will imply that in a game with strategic complementarities, players' best response correspondences will be monotone nondecreasing in the strategies of the other players; hence any conclusion in the theory of supermodular games which relied solely on this feature will also be true in this larger class of games with strategic complementarities. However, Milgrom and Roberts (1990b) established certain results concerning the coincidence of solution concepts and stability under learning dynamics that depend not just on the monotonicity of the best response correspondences but also on the more detailed structure of these games. The main results of this section show that these conclusions are still valid for this larger class of games. Stating the main results requires recalling a definition: the *serially undominated* strategies in a game are those that remain after iterated elimination of pure strategies that are strictly dominated by other pure strategies.

THEOREM 12: *Let Γ be a game with strategic complementarities. Then $\forall n \in N$, there exist strategies x_{*n} and x_n^* which are the smallest and largest serially undominated strategies for player n. Moreover, the pure strategy profiles $x_* \equiv (x_{*n}; n \in N)$ and $x^* \equiv (x_n^*; n \in N)$ are Nash equilibria.*

To prove Theorem 12, more notation and a lemma are required. Given $x \in S$, let $B_{*n}(x)$ denote the smallest best response of player n to x_{-n}, and $B_n^*(x)$ denote the largest best response to x_{-n} in a game with strategic complementarities, which are well-defined by the results of the previous sections and the assumptions of continuity and compactness. Let $B_*(x) \equiv (B_{*n}(x); n \in N)$ and $B^*(x) \equiv (B_n^*(x); n \in N)$. For $T \subset S$, define

$$U_n(T) \equiv \left\{ x_n \in S_n \mid \forall x_n' \in S_n, \exists \hat{x} \in T \text{ such that } \pi_n(x_n, \hat{x}_{-n}) \geqslant \pi_n(x_n', \hat{x}_{-n}) \right\}.$$

Then $U_n(T)$ represents the set of strategies of player n that are not strongly

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dominated when the player plays against strategies in T. Let $U(T) \equiv (U_n(T); n \in N)$ and $\bar{U}(T) \equiv [\inf\{U(T)\}, \sup\{U(T)\}]$.

LEMMA: *Let Γ be a game with strategic complementarities and $z_*, z^* \in S$ be such that $z_* \leqslant z^*$. Then* $\sup\{U([z_*, z^*])\} = B^*(z^*)$ *and* $\inf\{U([z_*, z^*])\} = B_*(z_*)$; *equivalently*, $\bar{U}([z_*, z^*]) = [B_*(z_*), B^*(z^*)]$.

PROOF: By definition, $B^*(z^*), B_*(z_*) \in U([z_*, z^*])$, hence

$$[B_*(z_*), B^*(z^*)] \subset \bar{U}([z_*, z^*]).$$

For the converse, we must show that if $z \notin [B_*(z_*, B^*(z^*)]$, then $z \notin U([z_*, z^*])$. There are two cases, according to whether $z \nleqslant y^* \equiv B^*(z^*)$ or $z \ngeqslant y_* \equiv B_*(z_*)$.

Suppose for some n, $z_n \nleqslant y_n^*$ and let $x \in [z_*, z^*]$. If $\pi_n(z_n, x_{-n}) - \pi_n(z_n \wedge y_n^*, x_{-n}) \geqslant 0$, then by the single crossing property, $\pi_n(z_n, z_{-n}^*) - \pi_n(z_n \wedge y_n^*, z_{-n}^*) \geqslant 0$. Then, by quasisupermodularity, $\pi_n(z_n \vee y_n^*, z_{-n}^*) - \pi_n(y_n^*, z_{-n}^*) \geqslant 0$. But $z_n \vee y_n^* > y_n^*$ (because $z_n \nleqslant y_n$), so by the definition of y_n^*, $\pi_n(z_n \vee y_n^*, z_{-n}^*) - \pi_n(y_n^*, z_{-n}^*) < 0$, a contradiction. Therefore, $\pi_n(z_n, x_{-n}) - \pi_n(z_n \wedge y_n^*, x_{-n}) < 0$. Thus $z_n \wedge y_n^*$ strongly dominates z_n against every $x \in [z_*, z^*]$. Hence $z \notin U([z_*, z^*])$.

A similar argument shows that if $z_n \ngeqslant y_n^*$ for some n, then $z_n \vee y_n^*$ strongly dominates z_n against strategies in $[z_*, z^*]$.

Therefore $\bar{U}([z_*, z^*]) = [B_*(z_*), B^*(z^*)]$. *Q.E.D.*

The proof of Theorem 12 proceeds exactly as the proof of the analogous result in Milgrom and Roberts (1990b), relying solely on the preceding lemma, the monotonicity of the operator U, the definition of serially undominated strategies, and continuity. See Shannon (1990) for additional details.

The results of this extension have significance beyond game theory. For the application to general equilibrium theory, we extend two more of the theorems of Milgrom and Roberts (1990b) to the larger class of games with (ordinal) strategic complementarities. The first is an equilibrium comparative statics proposition.

THEOREM 13: *Let $\Gamma_t = \{N, (S_n), \pi_n(x_n, x_{-n}, t)_{n \in N}\}$ be a family of games with strategic complementarities such that $\pi_n(x_n, x_{-n}, t)$ satisfies the single crossing property in $(x_n; x_{-n}, t)$ for all $n \in N$. Then the largest and smallest pure strategy equilibria (and serially undominated strategy profiles) $x_n^*(t)$ and $x_n^*(t)$, are monotone nondecreasing functions of the parameter t.*

The second concerns the stability of adaptive learning processes. In intuitive terms, a sequence is consistent with adaptive learning by the players if they eventually abandon strategies that perform consistently badly in the sense that there exists some other strategy that performs strictly and uniformly better against every combination of what the competitors have played in the not too

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distant past. Formally, a sequence $\{x(t)\} \subset \prod_{n \in N} S_n$ is *consistent with adaptive learning* if for all $\varepsilon > 0$ and all dates T there is some later date T' such that for all $t > T'$ and all strategies z_n and $z'_n \in S_n$, if $\pi_n(z_n, x_{-n}(s)) + \varepsilon < \pi_n(z'_n, x_{-n}(s))$ for all $s \in [T, t]$ then $x_n(t) \neq z_n$.

THEOREM 14: *For finite games with strategic complementarities, if $\{x(t)\}$ is consistent with adaptive learning, then there is some date T after which $x_* \leqslant x(t) \leqslant x^*$. For finite or infinite games with strategic complementarities, if the pure Nash equilibrium is unique, then a sequence $\{x(t)\}$ is consistent with adaptive learning if and only if it converges to the unique equilibrium.*

In Section 3, we established that the payoff functions in certain differentiated product Bertrand oligopoly models satisfy the single crossing property in $(p_n; p_{-n})$. The functions are also quasisupermodular in p_n, because the choice variable is one-dimensional. So Theorems 12–14 apply.

A second application of the theorems is a game derived from the general equilibrium model with gross substitutes. By applying the theorems to this game, we can quickly prove the existence and uniqueness of equilibrium, improve the previous best known stability results, and derive the main comparative equilibrium results for that class of models.

Consider an economy with $L + 1$ goods. The excess demand function for good n is written $d_n(p_n, p_{-n})$ and is assumed to be homogeneous of degree zero. Arrow and Hurwicz (1958) and Arrow, Block, and Hurwicz (1959) established that certain continuous-time, smooth tatonnement-like processes are globally stable price adjustment processes if the economy is characterized by gross substitutes, that is, if d_n is continuous and decreasing in p_n and continuous and monotone nondecreasing in p_{-n} for every n.

A fictional game can be constructed from this economy as follows. Fix one good, say good 0, to be numeraire. Let there be a market maker for each other good n who announces a price for that good. The market maker's payoff is defined to be $\pi_n(p_n, p_{-n}) = -|d_n(p_n, p_{-n})|$; that is, the market maker wants to come as close as possible to clearing the market. When the economy exhibits gross substitutes, it is routine to verify that the payoff function has the single crossing property. Several important conclusions follow.

First, using Theorem 12, one may reestablish the well known result that an equilibrium exists and is unique. If there were multiple equilibria, there would exist an equilibrium with the highest prices $\bar{p}$ and one with the lowest prices $\hat{p}$, where $\bar{p} > \hat{p}$. But the market for the numeraire good could not clear in both cases, since its demand is decreasing in these prices.

Second, in view of Theorem 14, any process consistent with adaptive learning by the individual market makers converges to the competitive equilibrium. This applies not just to continuous-time processes nor just to processes that use only current demand information. Both restrictions were imposed in the previously cited studies.

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Finally, one can use Theorem 13 to answer comparative statics questions about the equilibrium. Suppose that for some consumer j, all the goods are normal, and consider the effect of increasing consumer j's endowment of the numeraire good. Indeed, let the parameter t refer to consumer j's endowment of the numeraire and let $\pi_n(p_n, p_{-n}; t) = -|d_n(p_n, p_{-n}; t)|$. By definition of normality, all the non-numeraire demands d_n rise with t, so we have a parameterized family of games with the single crossing property. It follows from Theorem 13 that the equilibrium price vector is a nondecreasing function of t. Since the choice of numeraire is almost arbitrary (the numeraire must not be in excess supply at equilibrium), the same kind of conclusion applies for each of the goods.

6. CONCLUDING REMARKS

The theory of monotone comparative statics, even for optimization models, is still unfinished. One priority is the analysis of economic applications involving stochastic models. The first and second-order stochastic dominance relations, Blackwell's informativeness order for information systems, and the likelihood ratio order are among ones that could be usefully integrated with the single crossing and/or supermodularity conditions. Notions of "more correlated," which have proved relevant in welfare economics, have also been connected to supermodularity (Meyer and Mookherjee (1987)), but the connections are incomplete.

The comparative statics of equilibrium models is another subject ripe for study using the ordinal approach. Initial efforts in this direction have been made by Milgrom and Roberts (1992) and Villas-Boas (1992). Our own recent work (Milgrom and Shannon (1992)) applies the ordinal approach to study another equilibrium concept, exploring the structure and comparative statics of the core in a class of cooperative games.

Although we have focused on the standard regression-type comparative statics conclusions developed in this paper, multivariate comparative statics may also be important as an explanation of endogenous covariation in economic models. Indeed, if all of the components of a vector of endogenous variables x are nondecreasing functions of the same vector of independent shocks θ, then the endogenous variables will be positively correlated, and in fact will satisfy the more demanding statistical relationship called *association*. This idea has been applied with some success to explain the clustering of certain attributes in the theory of the firm (Milgrom and Roberts (1990a), Holmstrom and Milgrom (1992)) and seems a natural candidate for building and interpreting models of macroeconomic comovements across the business cycle.

Dept. of Economics, Stanford University, Stanford, CA 94305-6072, U.S.A.
and
Dept. of Economics, University of California—Berkeley, Berkeley, CA 94720, U.S.A.

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APPENDIX: Existence of a Maximum

So far, we have ignored the issue of existence of maximizers. In a finite dimensional setting, the assumptions of continuity and compactness are not overly restrictive and are relatively easy to check. In an infinite dimensional setting, however, the question of choice of topology is a more serious one. For our theory, the key is to connect the order-theoretic notions of completeness, quasisupermodularity, and convergence along chains with the topological notions of compactness and continuity. What links these ideas is the *order interval topology*, which is the topology resulting from taking the order intervals $\{[a,b],[a,+\infty),(-\infty,a]\}$ of the space as a sub-basis for the closed sets of the topology. The two key theorems are due to Frink (1942) and Veinott (1989).

Theorem A1 (Frink): *A lattice X is complete if and only if it is compact in the order interval topology.*

Theorem A2 (Veinott): *Let $\{S_\tau\}$ be a net of nonempty sets that is weakly ascending, that is, such that if $\tau' \geqslant \tau$, and $x \in S_\tau$, $x' \in S_{\tau'}$, then either $x \vee x' \in S_{\tau'}$ or $x \wedge x' \in S_\tau$. Then there exists a monotone selection $\{x(\tau)\}$ from $\{S_\tau\}$.*

To apply Theorem A2, observe that if f: $S \to \mathbb{R}$ is quasisupermodular, then $L_a \equiv \{x \mid f(x) \geqslant a\}$ is weakly ascending in a. The following theorem establishes that for quasisupermodular functions, upper semi-continuity along chains implies upper semi-continuity in the order interval topology.

Theorem A3: *If f: $X \to \mathbb{R}$ is quasisupermodular and if for every chain $C \subset X$, $\limsup_{x \in C, x \uparrow \sup(C)} f(x) \leqslant f(\sup(C))$ and $\limsup_{x \in C, x \downarrow \inf(C)} f(x) \leqslant f(\inf(C))$, then f is upper semi-continuous in the order interval topology.*

Proof: Let $S \equiv \{x \in W \mid f(x) \geqslant a\}$, and suppose the net $\{x_\tau\} \to x^*$. It suffices to show $f(x^*) \geqslant a$. Define $S_{\beta,\tau} \equiv S \cap \{x \mid \inf_{a \geqslant \beta} x_\alpha \leqslant x \leqslant \sup_{a \geqslant \tau} x_a\}$. By Veinott's theorem, for τ fixed, there exists a selection $y(\beta,\tau)$ which is monotone nondecreasing in β. Let $y(\tau) = \limsup y(\beta,\tau)$. Since $f(y(\beta,\tau)) \geq a$, by the condition of upper semi-continuity along chains, $f(y(\tau)) \geqslant a$. Let $T_\tau = S \cap \{x \mid x^* \leqslant x \leqslant \sup_{\alpha \geqslant \tau} x_\alpha\}$. Since $y(\tau) \geqslant x^*$, the sets T_τ are nonempty. By Veinott's theorem, there is a nonincreasing selection $\{z_\tau\}$ from $\{T_\tau\}$, with $f(z(\tau)) \geqslant a$. Then $z(\tau) \cdot x^*$, so upper semicontinuity along chains implies that $f(x^*) \geqslant a$. *Q.E.D.*

Combined with Frink's theorem and the second corollary of the monotonicity theorem, this theorem says that a quasisupermodular function on a complete sublattice that is upper semicontinuous along chains is upper semicontinuous in the order interval topology and hence has a nonempty, compact set of maximizers. Since the set of maximizers is also a sublattice, Frink's theorem implies that it is a complete sublattice, and hence has a greatest and least element. These observations are summarized in the following theorem.

Theorem A4: *Suppose that f: $X \to \mathbb{R}$ is quasisupermodular, that for every chain $C \subset X$, $\limsup_{x \in C, x \uparrow \sup(C)} f(x) \leqslant f(\sup(C))$ and $\limsup_{x \in C, x \downarrow \inf(C)} f(x) \leqslant f(\inf(C))$, and that S is a complete sublattice of X. Then $\arg\max_{x \in S} f(x)$ is a nonempty, complete lattice.*

The usefulness of these theorems lies in the fact that, for infinite dimensional spaces, it may often be easier to check that a function is upper semi-continuous along chains, and that its domain is a complete lattice, than to determine whether it is continuous and its domain compact in some appropriate topology. For example, the well-known results that for $1 \leqslant p < \infty$, intervals $[a,b]$ in L_p are complete lattices and that L_p-continuous functions are continuous in the order interval topology (see, e.g., Aliprantis and Burkinshaw (1985)) give rise to the conclusion that L_p-continuous, quasisupermodular functions achieve a maximum on any L_p-interval $[a,b]$. In contrast, the maximum does not necessarily exist if one replaces quasisupermodularity by concavity.

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