# *Optimal Sequential Search Among Alternatives*

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**Abstract**

We explore costly sequential search among finitely many risky options, and an outside option. *Payoffs are the sum of a known and hidden random factor.*

($a$) We resolve a long open question about how riskier payoffs impact search duration: *expected search time is higher for more dispersed idiosyncratic noise.*

($b$) Since options differ ex ante, we incorporate *selection effects* into search: Counterintuitively, with few options, the quitting chance falls if search costs rise; also, *while stopping rates rise over time, earlier options are recalled more.*

($c$) We find that the *stationary search model is a misleading benchmark*: For as the number of options explodes, the recall chance is bounded away from zero if the known factor has a distribution without a thin tail (eg. exponential).

($d$) A special case of our model captures *web search* engines that rank order options: We prove that the *click through rate* — the chance of initiating a search — is a poor quality measure since it falls in accuracy for expensive goods.

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# Contents

# 1 Introduction

We develop a general theory of search among finitely many options. Solving an open question, we determine when search duration rises in risk. By positing a tractable new model where option payoffs additively reflect hidden and observed random factors, we allow for the analysis of selection effects. This alters basic search theory insights — for example, recall of options may persist in the infinite horizon limit. A special case of our model yields a tractable framework of web search with new insights.

Our paper enhances the pure theory of sequential search, which has not advanced much since Weitzman's influential 1979 search model.[1] Despite a nonstationary search model with finitely many known distinct sampling distributions, Weitzman precisely characterized which current or prior option should next be searched using a simple index rule. Quite unlike stationary search, he found that one optimally sometimes wishes to recall a previously explored option. While both elegant and conceptually innovative, Weitzman's model offered few other new behavioral predictions.

We reformulate his framework to address this shortcoming. We introduce and fully solve a nonstationary sequential search model that yields a rich set of comparative statics, that is also general enough for a range of applications. We assume one known *outside option*, and finitely many *inside options*; payoffs are the sum of independent random *known* and *hidden* factors, drawn from logconcave distributions. *Ex post*, our known factors are fixed, and this yields a special case of Weitzman (1979). Here, a searcher optimally rank orders options by their known factors, and then explores them in this order, until either quitting search, recalling a previously explored option, or exercising the current option. Our additive two factor formulation sheds light on the probabilities of these three stopping events. Notably, we derive the first comparative statics for recall probabilities and for search duration in sequential search theory.

Since our known factors are random, we also generalize Weitzman (1979), *ex ante*. For his sampling payoff distributions are themselves random objects in our model. The resulting search model is ideal for estimation — as consumer preferences and information about products searched are unobservable to econometricians. Since we prove that options are explored in order of the known factor orders, we characterize search behavior ex ante, unconditional on preferences or information. Our predictions are driven by selection effects that so far have not been well-captured in search theory.

A final special case offers a tractable model of web search. We assume that before

[1]Two notable exceptions are Olszewski and Weber (2015), who find a more general index rule, and Doval (2013) who allows the searcher to freely exercise an unexplored option.

any ranking, option payoffs are independent Gaussian random variables. The search engine then exploits its knowledge of web pages and user cookies (e.g. the PageRank algorithm) to parse option payoffs into predicted (known) and idiosyncratic (hidden) factors — and a more accurate search engine has a lower idiosyncratic variance and higher predicted variance. Finally, it presents web pages to the user in order of decreasing known factors. In this story, the user quits if he exercises his outside option.

With stationary search, the plot of the value function is initially constant at the reservation prize, and then coincides with the $45^o$ diagonal. This reflects how the searcher surely rejects lower prizes and accepts larger prizes. In our world with recall, this classic value plot is replaced by a strictly convex value function, whose slope is strictly rising and equals the probability of eventually recalling the best prize seen (Figure 2). In other words, our nonstationary search model reveals a dual relationship between values and recall rates in harmony with the stationary search model.

This best-so-far is the state variable for the searcher's nonstationary dynamic programming problem. Search stops when the best prize so far exceeds Weitzman's index, which captures future optionality. Early on, the index is high, and the searcher chooses between the current option and further search. At some point, as the index continues to fall, the optimal choice is to accept the current option or recall. To wit, we prove that recall is exclusively a valuable option later in search.

In our first key contribution, we resolve the effect of a riskier prize distribution on search behavior. While it is well-known that risk unambiguously profits the searcher, its behavioral impact has long been ambiguous — even for stationary search, as highlighted long ago in Mortensen (1987). For a riskier prize distribution not only encourages more aggressive search, but also increases the weight in the favorable prize tail. As outlined in §2, the *dispersive order* — a ranking neither stronger nor weaker than the standard mean preserving spread — offers sharp behavioral predictions for search theory. Distributions rise in this partial order if all pairs of percentiles grow farther apart. Specifically, *the expected search time is higher and the quitting chance lower given a more dispersed idiosyncratic noise* (Theorems 1 and 3). Notably, dispersion in our known factor has roughly the opposite effect: Unless outside options are too large, the individual is more willing to initiate search and less willing to continue searching with greater dispersion in the known factor (Theorems 2).

The second contribution is the introduction of *selection bias*. For since the unobserved known factors are correlated top order statistics, the willingness to search or explore the next inside option signals the known factors. In principle, this might reverse intuitive comparative statics. For instance, one typically expects to search less, and quit more often, when search costs jump up. But anyone who still searches surely expects better inside options, and so is more likely to choose one. We argue that search costs still encourage quitting (unlike an extreme case explored in §2). But the selection effects intensify search over time, with *a rising hazard rate of recalling an earlier option and exercising a current one* (Theorem 4). In another manifestation of selection effects, the *options explored earlier are recalled more often* (Theorem 5). These are the first positive general results on recall probabilities in search theory.

Our third contribution exploits our tractable treatment of the number of options. While the infinite horizon model is the benchmark search model since McCall (1970), here it poorly approximates a large finite number $N$ of rank-ordered heterogeneous options. Theorem 6 finds that the hazard rate of recalling a prior option falls in the number of options, but need not vanish in the limit $N \to \infty$, and ignoring the recall option strictly hurts the searcher. The recall option is critically important when the known factor distribution lacks a "thin tail" — as with exponential payoffs. Only with a thin tail does the limit recall hazard rate intuitively vanish. So recall is an important search phenomenon, and stationary search is thus a misleading benchmark.

Our fourth major contribution is to *web search*, where a search means clicking on a link. For this sub-model, we assume that our two random factors are Gaussian random variables (and not merely log-concave). The chance of consuming web sites, possibly recalling them, increases each period. But a more accurate search engine yields opposing effects on search duration — it features more disperse known factors and less disperse idiosyncratic factors. Proposition 1 resolves the tradeoff, and finds that the quitting chance rises in accuracy for expensive goods, like furniture, and falls in accuracy for inexpensive goods like books, whereas the expected search time moves oppositely. The probability of initiating a search — the *click through rate* (CTR) — moves analogously to the expected search time. This means that its common usage as a ranking measure (eg. Experian Hitwise) is unjustified, since it is non-monotone in accuracy. Search duration might increase or decrease when accuracy improves.

We also can draw more normative conclusions for the web search environment. Inspired by Blackwell's Theorem, the user's value of the search engine information rises in the accuracy. In Proposition 2, we find that users with average search costs and outside options derive the greatest value from the search engine. We then ask how the search engine value changes in the number of options. In Proposition 3, we find a complementarity between the web size and the search engine accuracy, and a substitutability between the web size and search cost. In other words, as the web grows, the gains to a more accurate or easier to use search engine grow.

Literature. When individuals only learn the next known factor but not future known factors, our model is formally search with learning. Yet Rosenfield and Shapiro (1981) showed that the optimal price search strategy need not entail a cut-off rule, since expectations jump after prize draws. Our additive reward distribution escapes this pathological possibility. More subtly, we find that a log-concave distribution ensures that favorable prizes encourage further search — our selection effect.

Optimal sequential search models are sometimes used in the empirical industrial organization. Kim et al. (2010) estimates Weitzman's model with shopping data from Amazon, and De Los Santos et al. (2013) estimate a sequential consumer search model with learning. We complement this literature by fully characterizing optimal search behavior given search costs, etc. But lately, empirical work has questioned the applicability of non-sequential search models for web search. For instance, Dinerstein et al. (2014) apply fixed sample size search models to study a search redesign by eBay. Earlier, De Los Santos et al. (2012) studied the online market for books and argued that fixed sample size search models better explain the data. In §8.3, we show that selection effects allow us to rationalize their empirical findings in our sequential search environment — for intuitively, low prices encourage continued price search.

A new literature on equilibrium web search models explores price formation and market efficiency. For example, Baye and Morgan (2001) explore the equilibrium price dispersion arises in a market where buyers and sellers choose whether to adopt a search engine. Armstrong et al. (2009) and Armstrong and Zhou (2011) study the implications of consumer web search on sellers' pricing strategies. Choi et al. (2016) study an equilibrium version of our model where sellers post prices and buyers solve an optimal sequential search problem. Finally, Eliaz and Spiegler (2016) consider a two-sided matching problem with search frictions. They consider the optimal set of search results that a search engine should return to maximize matching efficiency.

## 2 Riskier Search and Selection Effects: A Foretaste

A. Risk. Consider for simplicity McCall's classic 1970 infinite horizon, job search model. Naturally there is a stationary solution characterized by a reservation wage. We explore how risk impacts search duration. With a discount factor $\beta$, wage $W \sim F$, and search cost $c$, the reservation wage is $\bar{w}(c) = -c + \beta[F(\bar{w}(c))\bar{w}(c) + \int_{\bar{w}(c)}^{\infty} w dF(w)]$.

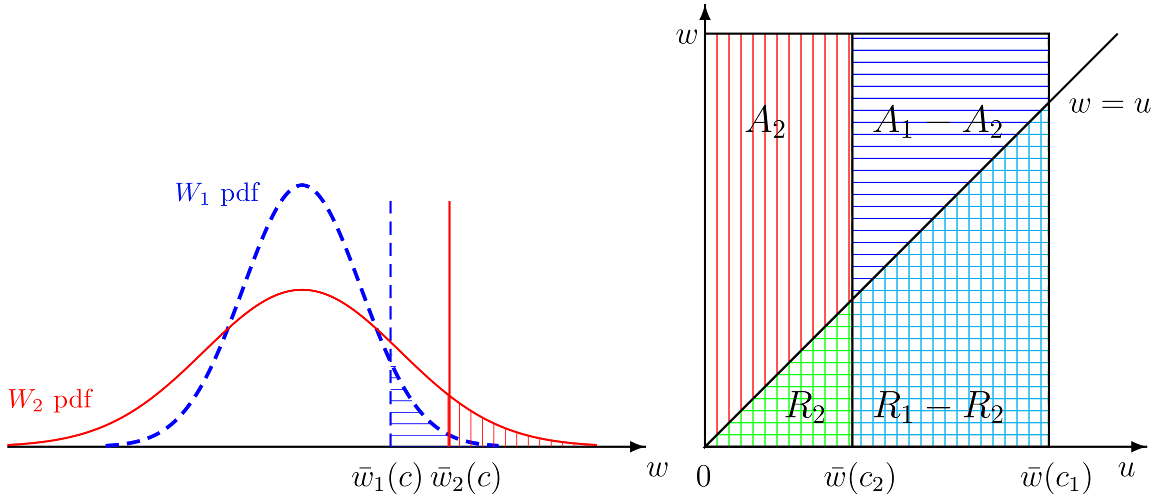


Figure 1: The left panel shows how a mean preserving spread in wages from $W_1$ to $W_2$ ambiguously impacts the stopping hazard rate $1 - F_i(\bar{w}_i(c))$ (shaded areas): For it pushes more mass to the tails, and raises the reservation wage $\bar{w}_i(c)$. The right panel depicts the selection bias in a one-stage job search problem with a random wage $W$ and random outside option $U$. The worker searches iff $U < \bar{w}(c)$, and accepts any job with wage $W \geq U$, i.e. above the diagonal line. As the search cost rises from $c_1$ to $c_2$, the reservation wage rises from $\bar{w}(c_1)$ to $\bar{w}(c_2)$. With unconditional probabilities $A_i$ and $R_i$ as labeled, the chance of accepting a job falls from $A_1/(A_1 + R_1)$ to $A_2/(A_2 + R_2)$.

Then

$$(1 - \beta)\bar{w}(c) = -c + \beta \int_{\bar{w}(c)}^{\infty} [1 - F(w)]dw. \tag{1}$$

The effect of a mean preserving spread of $W$ on the hazard rate of stopping $1 - F(\bar{w}(c))$ is ambiguous in general because there is more mass in the right tail of $F$ but $\bar{w}(c)$ also rises (left panel of Figure 1). We consider instead an order which globally spreads out probability weight. If $F_i$ is the cdf of $W_i$, then $W_2$ is *more dispersed* than $W_1$ if every pair of quantiles are further apart with $F_2$ than $F_1$. We claim a bifurcation: that *if $W$ grows more dispersed, then the stopping hazard rate rises for low search costs $c$, and falls for all higher $c$.* To see why, differentiate (1) to get $\bar{w}'(c) = -[1 - \beta + \beta(1 - F(\bar{w}(c)))]^{-1}$. Then

$$\frac{\partial[1 - F(\bar{w}(c))]}{\partial c} = -f(\bar{w}(c))\bar{w}'(c) = \frac{f(\bar{w}(c))}{1 - \beta + \beta(1 - F(\bar{w}(c)))}. \tag{2}$$

If $W_2$ is more dispersed than $W_1$, then whenever $F_2(\bar{w}_2(c)) = F_1(\bar{w}_1(c))$, we have $f_2(\bar{w}_2(c)) \leq f_1(\bar{w}_1(c))$. This single crossing logic yields the desired double inequality for the stopping hazard rate $[1 - F_2(\bar{w}_2(c))] \gtrless [1 - F_1(\bar{w}_1(c))]$ as $c \lessgtr \bar{c}$, for some real $\bar{c}$.

B. Selection Effects. Another major thrust concerns selection effects that emerge with heterogeneous options, when willingness to search signals a sufficiently promising wage distribution. For assume a random outside option payoff $U$ with only one search possible. So the worker searches iff $U < \bar{w}(c)$, and accepts any job paying $W > U$. As search costs rises from $c_1$ to $c_2 > c_1$, the reservation wage falls from $\bar{w}(c_1)$ to $\bar{w}(c_2)$. As seen in Figure 1, with $(W, U)$ independently and uniformly distributed, the search chance then falls from $A_1 + R_1$ to $A_2 + R_2$, the chance of accepting a job falls from $A_1$ to $A_2$, but this acceptance chance, conditional on search, *rises* from $A_1/(A_1 + R_1)$ to $A_2/(A_2 + R_2)$. So the selection bias reverses the effect of a higher search cost — as search cost rises, the worker searches less often and eventually accepts a job less often. But conditional on search, each search succeeds more often.

# 3 Model

A searcher must exercise exactly one option from $N < \infty$ *inside options* and one *outside option*. The latter has known payoff $u \in \mathbb{R}$. The payoff of an inside option is the sum $W = \mathcal{X} + \mathcal{Z}$ of a random *known factor* $\mathcal{X}$ and a random *hidden factor* $\mathcal{Z}$.

The factors $\mathcal{X}$ and $\mathcal{Z}$ have respective cdf's $G$ and $H$ (standing for "gnostic" and "hidden"). Their densities $g$ and $h$ are smooth and log-concave with full support on $\mathbb{R}$.[2] We assume *prospective independence*, namely, $\mathcal{X}$ and $\mathcal{Z}$ are jointly independent random variables. *Ex ante* probabilities are computed before $\mathcal{X}$ and $\mathcal{Z}$ are realized.

The searcher sees or learns the realized known factors $\chi$ prior to search; one can interpret it as an attribute that is unobserved by the modeler. He only learns the realized hidden factor $z$ when he *explores* the inside option. This incurs a *search cost* $c > 0$, and takes one stage. The searcher *participates* if he explores any inside option. After seeing its payoff, he may *pass* on an option, and explore another option. If the searcher participates, he may search in any order until any stage $n = 1, 2, \ldots, N$, and then exercise either the outside option or any inside option already explored. *Stopping* search entails *exercising* an option, namely accepting its payoff, thereby ending the search; this includes not participating. A searcher may exercise the current option — called *striking*, in the spirit of finance. Exercising a previously passed inside option is *recalling*. Exercising the outside option is called *quitting*.

In summary, if a searcher stops at *stage* $n = 0, 1, 2, \ldots, N$, then his net payoff is $w - nc$ if exercises an inside option with payoff $w$, and $u - nc$ if he quits. One can capture search with no outside option by assuming an outside option payoff $u = -\infty$.

[2]It is well-known (since Heckman and Honore (1990)) that log concave distributions are needed to ensure well-behaved behavior of expectation from truncated distributions. We make this assumption, keeping in mind that exploring an option entails deciding on the basis of a truncated distribution. Since $g$ is log-concave, so is $1 - G$, by Prekopa's Theorem, and $g/[1 - G]$ is non-decreasing.

# 4 Optimal Stopping Characterization

When the known factor has a degenerate distribution, say $\mathcal{X} = 0$, this model captures finite stage search from a fixed distribution; the searcher employs a constant cutoff, and only uses the recall option if he reaches the last period. When the hidden factor is constant, say $\mathcal{Z} \equiv \bar{z}$, the searcher perfectly sorts options, and stops at the first — to wit, no recall. But when $\mathcal{X}$ and $\mathcal{Z}$ have non-degenerate distributions, the searcher confronts a nontrivial nonstationary search problem, and sometimes recalls an option.

Indeed, rank order all options by realized known factors: $x_1 \geq x_2 \geq \cdots \geq x_N$. We argue that the searcher optimally explores options in this order. So if their realized payoffs are $w_1, w_2, \ldots, w_N$, then the dynamic programming state variable is the *best option so far*: $\Omega_0 = u$ and $\Omega_n = \max(u, w_1, w_2, \ldots, w_n)$ for stages $n = 1, 2, \ldots, N$.

Let $F_n$ denote the distribution of the random payoff $W_n = x_n + \mathcal{Z}_n$, corresponding to the option with known factor $\mathcal{X}_n = x_n$. Its cdf is thus $F_n(w) = H(w - x_n)$. Next, as in Weitzman (1979), implicitly define $n$ *reservation prizes* $\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N\}$ for the options:

$$\bar{w}_n = -c + \bar{w}_n F_n(\bar{w}_n) + \int_{\bar{w}_n}^{\infty} w dF_n(w). \tag{3}$$

Integration by parts yields $c = \int_{\bar{w}_n}^{\infty} 1 - F_n(z)dz$. As $x_n \geq x_{n+1}$, the distributions $F_n$ *stochastically fall* in $n$, or $F_{n+1}$ is FOSD below $F_n$. Then the reservation prizes fall each stage, namely, $\bar{w}_1 \geq \bar{w}_2 \geq \cdots \geq \bar{w}_N$. By Weitzman (1979), one explores options in order of reservation prizes, and thus known factors: One stops at stage-$n$ if the best-so-far option $\Omega_n$ exceeds the reservation prize $\bar{w}_{n+1}$ of exploring the next stage.

**Lemma 1** *The searcher optimally explores options in order $n = 1, 2, \ldots$, and stops searching at the first stage $n$ with $\Omega_n \geq \bar{w}_{n+1}$, accepting the best option so far.*

The *value function* $V_n(\Omega_n)$ at stage $n$ is the maximum payoff assuming optimal future behavior when the best option so far is $\Omega_n$. Clearly $V_N(\Omega_N) = \Omega_N$. For any $n < N$, backward induction logic recursively yields value functions $V_{n-1}, \ldots, V_1$ by:

$$V_n(\Omega_n) = \max\left\{\Omega_n, -c + V_{n+1}(\Omega_n)F_{n+1}(\Omega_n) + \int_{\Omega_n}^{\infty} V_{n+1}(z)dF_{n+1}(z)\right\}. \tag{4}$$

Subtly, this recursion assumes that the searcher foresees all future known factors. But since the reservation prize $\bar{w}_{n+1}$ depends only on $F_{n+1}$ and not $F_{n+2}, F_{n+3}, \ldots$ in (3), the searcher can always optimally stop at stage $n$ apprised only of the next known factor $x_{n+1}$. This reflects the one-stage look-ahead property of optimal search.[3]

[3] We appealed to Weitzman (1979). But a proof by the one-stage look-ahead property works, as the search problem is *monotone*, i.e. if a searcher stops at stage $n$ then he also stops at stage $n + 1$.

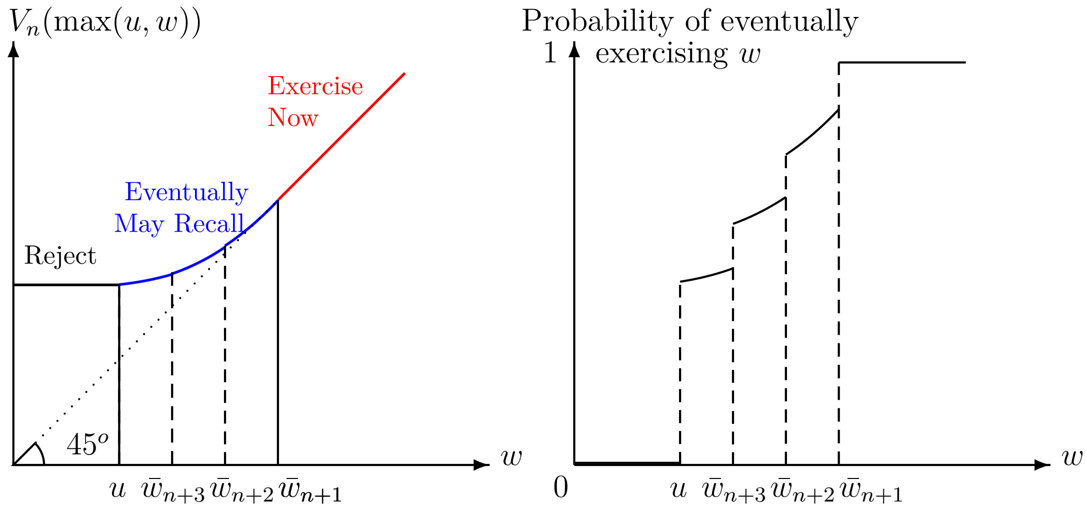


Figure 2: **Value Function and Recall Chances.** At left is the value $V_n$ at stage $n$ as a function of the best *inside option* so far $w = \max(w_1, \ldots, w_n)$. Payoffs $w \geq \bar{w}_{n+1}$ are exercised. Payoffs $w$ in $[u, \bar{w}_{n+1})$ are rejected, but may eventually be recalled. Payoffs $w < u$ are forever rejected. So $V_n$ is flat on $(-\infty, u)$, then increasing and strictly convex on $[u, \bar{w}_{n+1})$, then the $45^o$ diagonal. *Its slope is the chance of eventually exercising the current best-so-far payoff* $w$, seen at right for $\bar{w}_{n+3} > u > \bar{w}_{i+4}$.

**Lemma 2** *The stage $n = 1, \ldots, N$ value function $V_n$ is convex, and differentiable except at $\Omega = \bar{w}_N < \cdots < \bar{w}_{n+1}$, where $V_n'(\Omega+) > V_n'(\Omega-)$. The slope $V_n'$ increases on $[\bar{w}_n, \bar{w}_{n+1}]$, and is the chance of eventually exercising the stage $n$ best option so far $\Omega_n$. At stage $k = 2, \ldots, N$, an option $w$ is recalled iff $w \in [\bar{w}_{k+1}, \bar{w}_k)$ and $\Omega_k = w$.*

*Proof by Backward Induction:* All claims hold if $n = N$: $V_N(\Omega) = \Omega$ for $\Omega \geq \bar{w}_{N-1}$ as the best option $\Omega_N$ is exercised. Then $V_N'(\Omega) = 1$. Posit all claims at stage $n+1$. Search stops at stage $n$ if $\Omega_n \geq \bar{w}_{n+1}$. By (4), $V_n(\Omega) = \Omega$ on $[\bar{w}_{n+1}, \infty)$ and so $V_n'(\Omega) = 1$, i.e., the stopping chance. One searches at stage $n+1$ if $\Omega_n < \bar{w}_{n+1}$. Then $V_n'(\Omega_n) = F_{n+1}(\Omega_n) V_{n+1}'(\Omega_n)$ by (4). Since $V_{n+1}'$ jumps up at $\bar{w}_N < \cdots < \bar{w}_{n+2}$, so does $V_n'$. Now, $1 = V_n'(\bar{w}_{n+1}+) > V_n'(\bar{w}_{n+1}-) = F_{n+1}(\bar{w}_{n+1}) V_{n+1}'(\bar{w}_{n+1}-)$ as $V_{n+1}'(\bar{w}_{n+1}-) < 1$ by assumption, and $F_{n+1}(\bar{w}_{n+1}) < 1$. Then $V_n'$ exists except at jumps $\bar{w}_N < \cdots < \bar{w}_{n+1}$. If $\Omega_n < \bar{w}_{n+1}$, then the searcher enters stage $n+1$ and will recall $\Omega_n$ with chance $V_n'(\Omega_n) = F_{n+1}(\Omega_n) V_{n+1}'(\Omega_n)$. As $F_{n+1}$ has full support and $V_{n+1}$ is convex, $F_{n+1} V_{n+1}'$ rises, and $F_{n+1}(\Omega_n) < 1$. So $V_n$ is strictly convex and $V_n'(\Omega) < 1$ for all $\Omega < \bar{w}_{n+1}$. The last claim is true because the searcher enters stage $k$ with best-so-far $w$ iff $w = \Omega_{k-1} < \bar{w}_k$, and then he recalls $w$ iff $w = \Omega_k \geq \bar{w}_{k+1}$ by Lemma 1. $\square$

In stationary wage search, the value function is piecewise linear — first constant if the current wage lies below the reservation wage, and then the 45 degree line. As seen in Figure 2, the value $V_n$ in our model is increasing and strictly convex on an interval $[u, \bar{w}_{n+1}]$. The value increases in this interval since the best-so-far acts as insurance: The searcher recalls and exercises this option with positive probability. The value is strictly convex because a higher fallback option is recalled with a higher chance.

Naturally, when search is more costly, there is a higher chance that one exercises the current best-so-far option immediately or recalls it later. As a result, the value function in Figure 2 grows steeper as the search cost $c$ rises (§A in Online Appendix).

In our nonstationary model, recall acts as insurance: The searcher is strictly more ambitious than is justified by his continuous payoffs — i.e. his reservation prize $\bar{w}_{n+1}$ exceeds the continuation value $V_n$ — since he can recall past options. Without recall, reservation prizes agree with the continuation values, as in stationary wage search.

Capturing the benefits of the idiosyncratic noise $\mathcal{Z}$, the *search optionality value* $\zeta(c)$ implicitly solves:

$$c = \int_{\zeta(c)}^{\infty} [1 - H(z)]dz. \tag{5}$$

This is the Bellman equation for the reservation wage in stationary wage search (with $\mathcal{X} \equiv 0$). It falls in the search cost $c$, and rises if $\mathcal{Z}$ incurs a mean-preserving spread.

**Lemma 3 (Optimal Search)** *The stage $n$ reservation prize equals $\bar{w}_n = \chi_n + \zeta(c)$.*

This expression follows from integrating the tail probability (3) by parts, using (5):[4]

$$c = \int_{\bar{w}_n}^{\infty} [1 - F_n(s)]ds = \int_{\bar{w}_n}^{\infty} [1 - H(r - \chi_n)]dr = \int_{\bar{w}_n - \chi_n}^{\infty} [1 - H(z)]dz.$$

The searcher has three courses of action, depicted in Figure 3. By Lemmas 1 and 3, he strikes if $x_n + z_n \geq \bar{w}_{n+1}$ and $x_n + z_n \geq \Omega_{n-1}$, passes if $\bar{w}_{n+1} > x_n + z_n$ and $\bar{w}_{n+1} > \Omega_{n-1}$, and quits or recalls if $\Omega_{n-1} > x_n + z_n$ and $\Omega_{n-1} \geq \bar{w}_{n+1}$. Since the best-so-far $\Omega_n$ increases in $n$, while the future value $\bar{w}_n$ falls in $n$, by Lemma 2, we might end up with $\Omega_{n-1} \geq \bar{w}_{n+1}$, if we do not start there. So *early on, the searcher's choice is strike or pass*, since the future is brighter than the past, or $\bar{w}_{n+1} > \Omega_{n-1}$. *But eventually, the choice is either strike or quit / recall*, as $\Omega_{n-1} \geq \bar{w}_{n+1}$.

In the spirit of Weitzman (1979), one can solve the $n$ option search problem by solving a sequence of two option problems. Assume two random inside options $A$

[4] The additive expression relies on no discounting. While this assumption can be justified as natural for consumer search, since this typically lasts for a short period of time, this additive structure does not arise with a discount factor $\beta < 1$. For then (3) becomes $\bar{w}_n = \beta[-c + \bar{w}_n F_n(\bar{w}_n) + \int_{\bar{w}_n}^{\infty} w dF_n(w)]$. So if $\bar{w}_n = x_n + \zeta(c, x_n)$, then $\zeta$ solves $[x_n + \zeta](1-\beta)/\beta + c = \int_{\zeta}^{\infty}[1 - H(s)]ds$. If $\beta \in (0,1)$, then $\zeta(c, x_n)$ falls in $x_n$, invalidating the additive expression in Lemma 3.

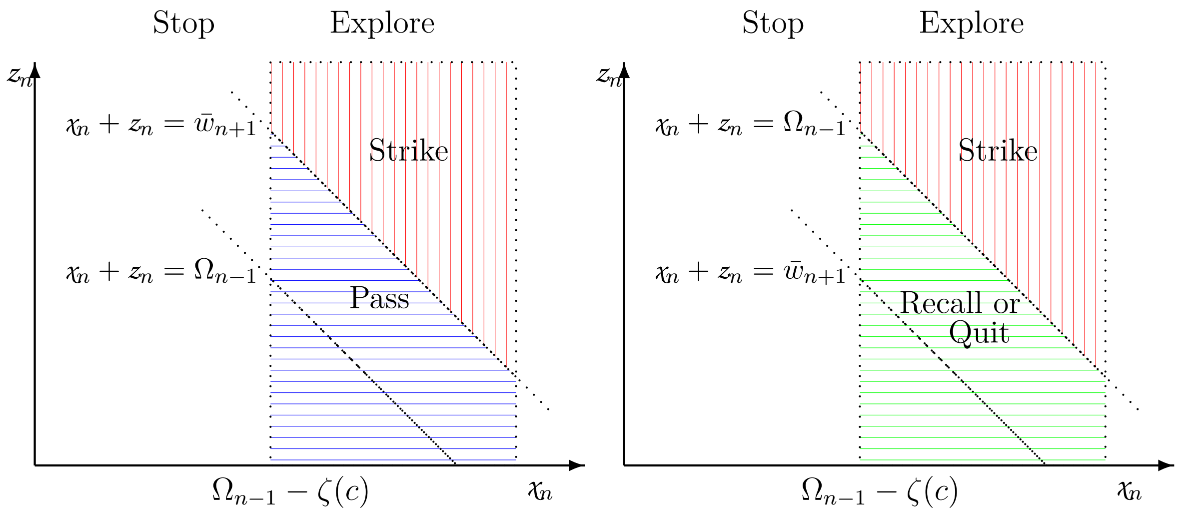


Figure 3: **Optimal Stopping Early and Late.** We plot behavior as a function of the known and idiosyncratic factors, $\chi$ and $z$. Early on, when $\bar{w}_{n+1} > \Omega_{n-1}$, the searcher always decides between strike and pass (left). But we eventually transition to $\Omega_{n-1} \geq \bar{w}_{n+1}$, whereupon the decision margin shifts to strike or recall / quit (right).

and $B$ and no outside option. We say that $A$ *delays* $B$ if $\bar{w}_A > \bar{w}_B$ and $w_A < \bar{w}_B$. For then the searcher first explores $A$ and finds that its payoff $w_A$ is below $\bar{w}_B$, and so then explores option $B$. The *delay chance* $\delta(\chi, c)$ is the probability that an option $A$ with random known factor $\mathcal{X}$ delays an option $B$ with a fixed known factor $\chi$. So:

$$\delta(\chi, c) \equiv P\left(\{\mathcal{X} > \chi\} \cap \{\mathcal{X} + \mathcal{Z} < \chi + \zeta(c)\}\right) = \int_{\chi}^{\infty} H\left(\chi + \zeta(c) - r\right) g(r) dr. \quad (6)$$

The *attraction* $\pi(\chi, c)$ is the probability that option $B$ with fixed known factor $\chi$ is explored before option $A$ with random known factor $\mathcal{X}$, i.e., $\bar{w}_B > \bar{w}_A$ or $A$ delays $B$:

$$\pi(\chi, c) \equiv P(\{\mathcal{X} < \chi\} \cup \{\{\mathcal{X} \geq \chi\} \cap \{\mathcal{X} + \mathcal{Z} < \chi + \zeta(c)\}\}) = G(\chi) + \delta(\chi, c). \quad (7)$$

These two probabilities are key. For assume only two inside options $A$ and $B$, for simplicity. Then the probability that both options are explored — namely, the chance of the event that $A$ delays $B$ or that $B$ delays $A$ — is the sum of the (expected) delay chances. Equally well, the expected number of options explored is the sum of the chances that $A$ and $B$ are explored, namely, the sum of their (expected) attractions.

We next explore how predictions of our model change with its basic parameters. In §5, we discover a new and general way that risk impacts search duration. In §6, we characterize how search changes over time. And in §7, we ask whether the stationary search benchmark model corresponds to the limit with infinitely many options.

# 5 Prize Dispersion and Sequential Search

Now we derive the impact of a more dispersed reward distribution on search duration. We find that the effect of more dispersed known and idiosyncratic factors is opposite.[5]

## 5.1 Dispersion Impacts Search Duration and Participation

We develop a formula for the *survival chance* $\sigma_n$, namely, the probability that search lasts for at least $n$ stages. Most easily, $\sigma_0 = 1 > 0 = \sigma_{N+1}$, while $\sigma_1$ is the *participation chance*, i.e., the ex ante probability of initiating the search. Now, consider any $\sigma_n$, for $n = 1, \ldots, N$. This is the chance that $(a)$ $n-1$ options delay option $n$, namely $\mathcal{X} > \chi$ and $\mathcal{X} + \mathcal{Z} < \chi + \zeta(c)$, and $(b)$ the other $N-n$ options have a known factor below $\chi$, across all known factors $\chi > u - \zeta(c)$. Events $(a)$ and $(b)$ have probabilities $\delta(\chi, c)^{n-1}$ and $G(\chi)^{N-n}$. Since each option is the $n$th best overall with the same ex-ante chance, by prospective independence, integrating the binomial probability of events $(a)$ and $(b)$ over all possible known factors $\chi$ and all $n$ options yields the survival chance formula:

$$\sigma_n = N\binom{N-1}{n-1}\int_{u-\zeta(c)}^{\infty} \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi) d\chi. \tag{8}$$

*The survival chance $\sigma_n$ falls in the search cost $c$* since the cutoff known factor $u - \zeta(c)$ rises in $c$ by (5), and the delay chance $\delta(\chi, c)$ falls in $c$ by (5)–(6). More easily, *the survival chance $\sigma_n$ falls in the outside option payoff $u$* as $u - \zeta(c)$ rises in $u$.

Express the *duration* — the expected search time until stopping — as $\tau = \sum_{n=1}^{N} \sigma_n$. The survival chances and so duration fall in the outside option payoff $u$ and search cost $c$. By (8) and the attraction formula $\pi(\chi, c) = G(\chi) + \delta(\chi, c)$ in (7):

$$\tau = N\int_{u-\zeta(c)}^{\infty} \sum_{n=1}^{N} \binom{N-1}{n-1} \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi) d\chi = N\int_{u-\zeta(c)}^{\infty} \pi(\chi, c)^{N-1} g(\chi) d\chi. \tag{9}$$

Using this search duration formula, we now explore the ambivalent impact of a riskier prize distribution: For the searcher grows more ambitious, but there is more weight in each *tail*.[6] To analyze risk here, assume first standard stationary search with

[5] Ganuza and Penalva (2010) use the dispersive order to study information disclosure in auctions. Zhou (2016) and Choi et al. (2016) use it to study pricing in discrete choice models. In their paper on notions of risk, Chateauneuf et al. (2004) include an example of stationary search with no discounting, in which duration rises if the reward distribution grows *location independent riskier* (Jewitt (1989)).

[6] A left or right tail of the cdf has weight below 0.5. Eg. Let $P(\mathcal{Z}_1 = 1) = P(\mathcal{Z}_1 = 2) = P(\mathcal{Z}_1 = 3) = 1/3$. Let $c = 1/4$. So $\zeta_1(c) = 3$ by (5). The stopping hazard rate is $P(\mathcal{Z}_1 \geq 3) = 1/3$. Let the hidden factor have a MPS so that $P(\mathcal{Z}_2 = 1) = P(\mathcal{Z}_2 = 3) = 1/2$. Now, $\zeta_2(c) = 3$ by (5), and $P(\mathcal{Z}_2 \geq 3) = 1/2$. The stopping hazard rate rises as $\mathcal{Z}$ grows riskier. But if $c = 1/2$, then $\zeta_1(c) = 2$

no outside option, so that $\mathcal{X} \equiv 0, n = \infty$, and $u = -\infty$. A mean-preserving spread of $\mathcal{Z}$ raises the search optionality value $\zeta(c)$, by (5), but raises the weight in the right tail. To resolve the net effect on the *stopping hazard rate* $\mathcal{S} = 1 - H(\zeta(c))$, change variables in the standard search Bellman equation (5) from the prize $z$ to its quantile $a$, via the *quantile function* $z = H^{-1}(a)$. Since $dz = dH^{-1}(a) = [\partial H^{-1}(a)/\partial a]da$, we have:

$$\int_{1-\mathcal{S}}^{1} (1-a)\frac{\partial H^{-1}(a)}{\partial a}da = \int_{\zeta(c)}^{\infty}[1 - H(z)]dz = c.$$

Given cdf's $H_1$ and $H_2$, the associated stopping hazard rates are ranked $\mathcal{S}_2 < \mathcal{S}_1$ if the quantile function $H_2^{-1}$ is globally steeper than $H_1^{-1}$, i.e., $\partial H_2^{-1}(a)/\partial a > \partial H_1^{-1}(a)/\partial a$ at all quantile levels $a \in (0, 1)$. A sufficient condition for this inequality is that $h_2(H_2^{-1}(a)) \leq h_1(H_1^{-1}(a))$ for all $a \in (0, 1)$, where $h_n = H_n'$ is the density of $\mathcal{Z}_n$, for $n = 1, 2$ — namely, that $\mathcal{Z}_2$ is *more dispersed than* $\mathcal{Z}_1$ (written $\mathcal{Z}_2 \succeq_{disp} \mathcal{Z}_1$).

The dispersive order suffices to rank stopping hazard rates in the stationary world in §2. But in our nonstationary model, a stronger ranking is needed. The random variable $\mathcal{Z}_2$ is a *mean-enhancing dispersion* of $\mathcal{Z}_1$ if $\mathcal{Z}_2 \succeq_{disp} \mathcal{Z}_1$ and $E[\mathcal{Z}_2] \geq E[\mathcal{Z}_1]$. It is a *mean-preserving dispersion* if $E[\mathcal{Z}_2] = E[\mathcal{Z}_1]$. This is stronger than saying that $H_2$ is a mean preserving spread of $H_1$. For since $H_2^{-1}$ is steeper than $H_1^{-1}$, while $H_1$ and $H_2$ have the same mean, $H_1$ single crosses $H_2$ (see Diamond and Stiglitz (1974)).

**Theorem 1 (Hidden Factors)** *A mean enhancing dispersion in $\mathcal{Z}$ raises survival chances $\sigma_n$, for $n = 1, 2, \ldots$, and so also the participation chance and search duration.*

Next, consider the known factor $\mathcal{X}$. By Lemma 3, a searcher participates for a sufficiently inviting best option: $\mathcal{X}_1 > u - \zeta(c)$. Naturally, if the random variable $\mathcal{X}$ stochastically rises (falls), then so does the best known factor $\mathcal{X}_1$, and the participation chance $\sigma_1$ rises (falls). Next, assume instead that $\mathcal{X}$ grows more dispersed but neither rises or falls stochastically. Its impact on $\sigma_1$ depends on the outside option. Intuitively, if $u$ is large, then $u - \zeta(c)$ exceeds the median of $\mathcal{X}_1$. As $\mathcal{X}$ grows more dispersed, all quantiles of $\mathcal{X}_1$ push apart, inflating quantiles above the median, and so raising the chance $\sigma_1$ that $\mathcal{X}_1 > u - \zeta(c)$. But if $u$ is small, then $u - \zeta(c)$ is below the median of $\mathcal{X}_1$, and the effect of dispersion reverses: It shrinks the bottom quantiles of $\mathcal{X}$, raising the chance of $\mathcal{X}_1 \leq u - \zeta(c)$, depressing $\sigma_1$. Part $(a)$ summarizes this:

**Theorem 2 (Known Factors)** $(a)$ *If $\mathcal{X}$ grows more dispersed, but neither rises nor falls stochastically, then the participation chance $\sigma_1$ falls for low outside options $u$,*

and $P(\mathcal{Z}_1 \geq 2) = 2/3$, while $\zeta_2(c) = 3$ and $P(\mathcal{Z}_2 \geq 3) = 1/2$. The stopping hazard rate drops.

*and rises for all higher outside options.* (*b*) *If $\mathcal{X}$ grows more dispersed and falls stochastically, then every survival chance $\sigma_n$ falls, as does the search duration.*

To understand part (*b*), note that dispersion in the known factor $\mathcal{X}$ has a different impact on the survival chance $\sigma_n$, and so search duration, than the participation rate. For the searcher stops sooner if its order statistics ($\mathcal{X}_n$) drop more rapidly. Since all gaps between order statistics $\mathcal{X}_n - \mathcal{X}_{n+1}$ increase when $\mathcal{X}$ grows more disperse, this depresses the survival chances $\sigma_n$, ceteris paribus. But inasmuch as dispersion raises participation (Theorem 2(a)), this by itself increases the survival chances. Part (*b*) resolves the ambivalence, by assuming that $\mathcal{X}$ stochastically falls; this eliminates the participation effect, and so the survival chance unambiguously falls with $\mathcal{X}$ dispersion.

## 5.2 Dispersion Impacts Quitting Chances

The *quitting chance* is the probability $q$ that a searcher either does not participate, or does, but eventually exercises his outside option. Conversely, he eventually elects an inside option with chance $1-q$. For instance, in online product search, after using a shopping search engine, a searcher eventually buys with chance $1-q$. In a stationary job search environment, if one is willing to search, then one never quits, and so $q = 0$.

The searcher explores option $j$ only if he does not quit before, and so if $\mathcal{X}_j + \zeta(c) \equiv \bar{w}_j > u$, by Lemma 3. So one explores option $j$ only if $\mathcal{X}_j > u - \zeta(c)$, which is thus a cutoff known factor. Option $j$ is *dominated* (by the outside option) if $\mathcal{X}_j \leq u - \zeta(c)$.

In a world with just one inside option, the *disappointment chance* is the probability that it is explored before the outside option is exercised. By Lemma 3, this probability equals the delay chance $\delta(u - \zeta(c), c)$, namely, for a hyothetical inside option with known factor payoff $u - \zeta(c)$. The quitting chance in a one-stage problem is the disappointment chance plus the nonparticipation chance, i.e., $\pi(u - \zeta(c), c)$ by (7).

Let $q_n$ be the *stage $n$ quitting chance*, namely: (1) the first $n$ options are explored and never exercised: $\mathcal{X}_j > u - \zeta(c)$ and $\mathcal{X}_j + \mathcal{Z}_j < u, \forall j \leq i$, and (2) later options are dominated: $\mathcal{X}_j \leq u - \zeta(c), \forall j > i$. Since an option is explored and never exercised with chance $\delta(u - \zeta(c), c)$, by prospective independence, the $n$ events have chance:

$$q_n = \binom{N}{n} \delta(u - \zeta(c), c)^n G(u - \zeta(c))^{N-n}. \tag{10}$$

Given the quitting chance $q = \Sigma_{n=0}^{N} q_n$, equations (7) and (10) yield a simple formula

$$q = \Sigma_{n=0}^{N} q_n = [\delta(u - \zeta(c)) + G(u - \zeta(c))]^n = \pi(u - \zeta(c), c)^n. \tag{11}$$

The quitting chance $q$ therefore geometrically falls in the number of options $n$.

**Theorem 3 (Hidden Factors)** *The quitting chance $q$ rises if $\mathcal{Z}$ incurs a mean preserving dispersion, for low outside options $u$, and falls for all higher outside options $u$.*

This corresponds to the effect of dispersion on the non-participation chance $1-\sigma_1$ (Theorem 2($a$)). A single option problem affords some quick intuition. For then, since the searcher quits if he does not participate ($\mathcal{X}+\zeta(c)<u$) or if he recalls the outside option ($\mathcal{X}+\mathcal{Z}<u$), one can rewrite the quitting chance in (11) with one option as a cdf of the random variable $\min(\mathcal{X}+\zeta(c),\mathcal{X}+\mathcal{Z})\equiv\mathcal{X}+\min(\mathcal{Z},\zeta(c))$, namely:

$$q=q_0+q_1=\pi(u-\zeta(c),c)=P(\mathcal{X}+\min(\mathcal{Z},\zeta(c))<u). \tag{12}$$

Easily, the quitting chance falls in the outside option $u$ and in the search cost $c$ since $\zeta'(c)<0$. A mean preserving dispersion of the hidden factor $\mathcal{Z}$ inflates the search optionality value $\zeta(c)$. This stochastically raises $\mathcal{X}+\min(\mathcal{Z},\zeta(c))$, and so shrinks (12). Moreover, the top quantiles of $\mathcal{Z}$ rise and the bottom quantiles fall. This is equally true of $\mathcal{X}+\min(\mathcal{Z},\zeta(c))$. Hence, as $\mathcal{Z}$ grows more dispersed, the quitting chance $q$ in (12) rises if $u$ is small, and falls for all large enough $u$. The result asserts more strongly that this comparative static is single crossing in $u$.

As the known factor $\mathcal{X}$ grows more disperse, $\mathcal{X}+\min(\mathcal{Z},\zeta(c))$ might not, and its effect on $q$ is unclear. Easily, the quitting chance $q$ falls as $\mathcal{X}$ or $\mathcal{Z}$ stochastically rises.

# 6 Selection Effects and Hazard Rates Over Time

We now see how selection effects arise with ranked known factors, and explore their counterintuitive consequences. Let $\mathcal{S}_n$, $\mathcal{Q}_n$, and $\mathcal{E}_n$ be the respective *hazard rates* of *stopping*, *quitting*, and *exercising* an inside option, as computed *ex ante*, conditional on entering stage $n$. Since the searcher either quits or exercises an inside option after stopping, $\mathcal{S}_n=\mathcal{Q}_n+\mathcal{E}_n$. Next, if $\mathcal{K}_n$ and $\mathcal{R}_n$ are the respective *hazard rates* of *striking* the current option or *recalling* an explored option at stage-$n$, then $\mathcal{E}_n=\mathcal{K}_n+\mathcal{R}_n$.

Our selection effects arise because search choices signal information about known factors: For one is more willing to explore option $n$ when search costs rise with a higher known factor $\mathcal{X}_n$, and so higher later factors $\mathcal{X}_{n+1}$, etc.

**Lemma 4** *Conditional on entering stage $n$, the factor $\mathcal{X}_n$ stochastically rises in the number $N$ of options, the search cost $c$, and the outside option $u$, and falls in $n$.*

We next explore how this selection effect impacts search hazard rates. Not only does the current known factor $\mathcal{X}_n$ increase in the cost $c$, but it rises more than $\mathcal{X}_{n+1}$, since $\mathcal{X}$ has a log-concave distribution. So the gap between known factors $\mathcal{X}_n - \mathcal{X}_{n+1}$ widens, and the survival probability $\sigma_n$ falls (as noted before §5.2). More strongly, we prove in §B.2 that $\sigma_{n+1}$ falls proportionately less than $\sigma_n$, raising the stopping hazard rate $\mathcal{S}_n \equiv 1 - \sigma_{n+1}/\sigma_n$. So one stops sooner with a higher search cost $c$, but this is more subtle with selection effects.[7]

Naturally, future options stochastically worsen each stage, since the known factors fall, while past inside fallback options grow more numerous. But opposing this is a selection effect: Conditional on arriving at a later stage, the fallback inside options are worse, and the outside option is more inviting. We show that the first effect dominates the second selection effect for $\mathcal{R}_n$ and $\mathcal{E}_n$, by log-concavity of $G$ and $H$.

**Theorem 4 (Search Intensifies)** *The recall and inside option exercise hazard rates, $\mathcal{R}_n$ and $\mathcal{E}_n$, rise in $n$. The quitting hazard rate $\mathcal{Q}_n$ rises for small search costs $c > 0$.*

For example, assume $\mathcal{X}$ and $\mathcal{Z}$ are Gaussian random variables with distribution $N(0, \alpha^2)$ and $N(0, 1-\alpha^2)$ respectively. If $(\alpha, c, u, n) = (0.4, 0.01, 1, 7)$, then $\mathcal{K}_n$ is U-shaped in $n$. If $(\alpha, c, u, n) = (0.4, 0.1^{-10}, 4, 5)$ then $\mathcal{Q}_n$ drops from stage 1 to stage 2 and then rises in $n$.

If $(\alpha, c, u, n) = (0.4, 1.2, -0.1, 7)$, the hazard rate of quitting falls in $n$ when $c$ or $u$ are very large.

So the probability of exercising any option, or any prior option, rises over time. While the recall probability rises, we now ask *which option one recalls.* While earlier options have larger known factors, they have been passed over more often; this offers more damning selection evidence of their hidden factors. Nevertheless:

**Theorem 5 (Older Options are Recalled More)** *If one explores option $n$, then the chance of recalling any prior option $j < n$ falls in $j$, for all $n = 1, 2, \ldots, N$.*

*Proof:* Since the searcher explores option $n$, the payoff of any prior option is less than the cutoff $\bar{w}_n = x_n + \zeta(c)$, or it would have been exercised earlier. By the Markov property of order statistics,[8] the joint distribution of the known and hidden

[7] Selection effects also surprisingly affect the *eventual quitting chance* $\bar{q} \equiv (q - q_0)/(1 - q_0)$, conditional on search. Intuitively, the quitting chance $q$ rises in $c$ (mentioned after (12)). But conditional on search, $\bar{q}$ falls in the search cost $c$ with a small number of options $N$ (Claim B.2).

[8] Theorem 2.4.1 (*The Markov Property*) in Arnold et al. (1992): Let $X_{1:n} \geq X_{2:n} \geq \cdots \geq X_{n:n}$ be order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a population with cdf $F$ and pdf $f$. Then given $X_{i:n} = x_i$, the distribution of $X_{j:n}$, for $j < i$, is the same as the distribution of the $j$-th order statistic of an $(n - i)$ sample from a population with distribution $F$ truncated at the left by $x_i$.

factors for the first $n-1$ options equals that of $n-1$ i.i.d. draws $(\mathcal{X},\mathcal{Z})$ from $(G,H)$, *conditional* on the known ranking $\mathcal{X} > \chi_n$ and the selection effect $\mathcal{X}+\mathcal{Z} < \chi_n + \zeta(c)$. If $\mathcal{X} = \chi > \chi_n$ is the realized known factor of any prior option, its payoff $W \equiv \chi + \mathcal{Z}$ has cdf

$$P(W < w | W < \chi_n + \zeta(c)) = \frac{H(w-\chi)}{H(\chi_n + \zeta(c) - \chi)}.$$

Since $w < \chi_n + \zeta(c)$, this falls in $\chi$ by log-concavity of $H$, and $W$ stochastically increases in $\chi$. Hence, the payoffs of earlier options are stochastically ranked. Because this ordering holds for all $\mathcal{X}_n$ realizations, it is also holds unconditional on $\mathcal{X}_n$. $\square$

# 7 Is Stationary Search a Good Benchmark?

We next explore how well the infinite horizon model approximates the actual long finite horizons that exist. In a stationary search model, one never quits or recalls — hazard rates are constant at $\mathcal{Q} = \mathcal{R} = 0$ — while striking occurs at a constant hazard rate $\mathcal{K} = 1 - H(\zeta(c))$. With a finite number $N$ of options, the quitting hazard rate $\mathcal{Q}_n$ falls in $N$. For upon entering stage $n$, the known factor $\mathcal{X}_n$ rises in $N$, by Lemma 4. Given correlated order statistics, both fallback and future options are more attractive too, deterring quitting. But the decision to quit also reflects the expected improvement in future options. Intuitively, the gaps $\mathcal{X}_n - \mathcal{X}_{n+1}$ between consecutive known factors stochastically shrink as $N$ increases. This spurs search, depressing hazard rates $\mathcal{K}_n$, $\mathcal{R}_n$ and $\mathcal{Q}_n$ as $N$ grows.

Next, consider the limiting behavior as $N$ explodes. One might guess that the predictions resemble those of a stationary search model since the gaps between known factors $\mathcal{X}_n$ vanish — i.e. the known factors are nearly constant. This intuitive guess is wrong: For the limit gaps $\mathcal{X}_n - \mathcal{X}_{n+1}$ depend on the right tail of the distribution $G$.

Since the hazard rate $g/[1-G]$ is non-decreasing, $\lim_{\chi \uparrow G^{-1}(1)} g(\chi)/[1-G(\chi)]$ exists and is positive. If $\lim_{\chi \uparrow G^{-1}(1)} g(\chi)/[1-G(\chi)] = \infty$, $G$ has a *thin tail*[9] — e.g. the uniform and Gaussian distributions have a thin tail and the exponential does not. This definition affords us a sharp characterization of limit search behavior:

**Theorem 6 (Hazard Rates and Number of Options)** *The hazard rates $\mathcal{Q}_n$, $\mathcal{R}_n$ and $\mathcal{K}_n$ of quitting, recalling, and striking in any stage $n$ all fall in $N$. As $N \to \infty$, for*

[9]Absent the thin tail of $G$, the right tail vanishes exponentially fast as $\chi$ explodes (Claim C.1). Also, the first order statistic of $n$ i.i.d. draws from $G$ is approximately Gumbel distributed for large $n$. This is a well-known result in the extreme value theory: *$G$ is in the domain of attraction of a Gumbel type general extreme value distribution.* See Leadbetter et al. (2012) for a comprehensive discussion.

*each stage $n$, $\mathcal{Q}_n$ vanishes, $\mathcal{R}_n$ vanishes if and only if $G$ has a thin tail, and $\mathcal{K}_n$ tends to $1-H(\zeta(c))\in(0,1)$ if $G$ has a thin tail, and otherwise has a strictly larger limit.*

Naturally, as the number of options $N$ explodes, the searcher never quits and strikes at a fixed rate. But *the limit recall hazard rate is positive if and only if the distribution $G$ of known factors does not have a thin tail.* In fact, to deny the recall option to the searcher strictly lowers his welfare in the limit $N \uparrow \infty$.[10] Moreoever, striking hazard rates should intuitively just reflect the hidden noise in the limit. But absent a thin tail, *one strikes in the limit more often than justified by hidden noise* — since the gaps between consecutive known factors do not vanish, and this provides an additional incentive to strike. All told, without thin tails, the infinite horizon model surprisingly offers a misleading approximation of large finite horizon search behavior.

Theorem 6 also implies that *search duration rises with more options.* For the striking hazard rate $\mathcal{S}_n \equiv 1 - \sigma_{n+1}/\sigma_n$ yields the survival chance formula $\sigma_n = \sigma_1 \Pi_{j=1}^{i}(1-\mathcal{S}_j)$. This rises in $N$, as $\sigma_1 = P(\mathcal{X}_1 > u - \zeta(c))$ rises in $N$ by Lemma 4, and $\mathcal{S}_j \equiv \mathcal{Q}_j + \mathcal{E}_j$ falls in $N$, by Theorem 6. So search duration $\tau \equiv \sum_{i=0}^{N} \sigma_n$ rises in $N$.

# 8 Application: Web Search

We now specialize our environment of known and idiosyncratic factors, to create a simple dynamic model web search. For we imagine that the search engine extracts known factors from web site payoffs, and uses them to pre-rank options for the user.

## 8.1 An Accuracy Model

A user can consume at most one web site from $n$ options. Each entails a *clicking cost* $c>0$. Not all is lost if the user does not find the good: The outside option is $u>-\infty$.

A search engine helps the user by sorting web sites according to his preference. After the user enters the search query, the search engine estimates his web site payoffs. The search engine then sorts the web sites in descending order of their expected payoffs, and posts a list of hyperlinks and short descriptions (revealing known factors). Seeing this list, the user clicks them sequentially. By Lemma 1, he need only learn the next known factor after each click. So the search engine orders the options exactly as the searcher would rank order them, thinking of these estimates as known factors.

[10] Before learning the known factor $\mathcal{X}_n$, given $G, H$ continuous, the searcher thinks the reservation prize $\bar{w}_{n+1}$ in Lemma 3 almost surely (a.s.) differs from the best-so-far $\Omega_n$, and so the searcher is a.s. not indifferent about stopping, by Lemma 1, and a.s. strictly prefers one option to all others.

Whereas so far we've assumed that the searcher rank-orders the options himself, here the search engine does it. We normalize the means of $\mathcal{X}$ and $\mathcal{Z}$ to zero, by adjusting the outside option. Greater accuracy means a lower hidden variance $E[\mathcal{X}^2]$ and a greater known variance $E[\mathcal{Z}^2]$, while leaving unchanged the distribution $\mathcal{X}+\mathcal{Z}$.

Write $\mathcal{X} = \alpha X$, and $\mathcal{Z} = \beta Z$, where $X$ and $Z$ each have unit variance, and are independent by prospective independence. But since $W \equiv \mathcal{X} + \mathcal{Z} \equiv \alpha X + \beta Z$ has a fixed distribution, its variance $\alpha^2 + \beta^2$ is constant. Normalizing $\alpha^2 + \beta^2 = 1$, we have $\beta = \sqrt{1-\alpha^2}$. Hence:

$$W = \alpha X + \sqrt{1-\alpha^2} Z. \tag{13}$$

*Accuracy* $\alpha$ transfers weight from the hidden to the known factor, for given $X, Z$. In fact, the distributions of $W, X$ and $Z$ coincide, since $W = Z$ when $\alpha = 0$ and $W = X$ when $\alpha = 1$. Since equation (13) holds for all $\alpha \in [0,1]$, their distribution is *stable*.[11] But the log-concavity of $\mathcal{X}, \mathcal{Z}$ implies a finite variance. Finally, $W, X$ and $Z$ are each Gaussian $N(0,1)$, as that is the only stable distribution with a finite variance.[12] As this setting is a special case of our original model, all earlier results in §4–§6 apply. Because scaling a distribution increases its dispersion,[13] the dispersion of the known factor rises and of the hidden factor falls as the accuracy $\alpha$ rises.

Aside from web search, our Gaussian accuracy model admits an interpretation in which greater accuracy arises from more precise Bayesian signals. Assume that the payoffs of inside options are Gaussian $W \sim N(0,1)$. Before searching, he observes a signal $X \sim N(\alpha w, \sqrt{1-\alpha^2})$ for each option with true value $w$ — say, a job advertisement. Upon seeing $X = x$, the searcher updates his posterior beliefs to $W \sim N(\alpha x, \sqrt{1-\alpha^2})$.[14] Since the noise in his estimate $Z = (W - \alpha x)/\sqrt{1-\alpha^2}$ is also $N(0,1)$, and is independent of $X$, the formula (13) arises with $X, Z \sim N(0,1)$.

Optimal stopping is governed by our model, where the optionality $\zeta(\alpha, c)$ solves a specialization of (5):

$$c = \int_{\zeta(\alpha,c)}^{\infty} \left[1 - \Phi\left(\frac{s}{\sqrt{1-\alpha^2}}\right)\right] ds. \tag{14}$$

where $\Phi$ is the Gaussian cdf. One participates in search iff $X_1 > \ell(\alpha, u, c)$, where

[11] By Definition 1.1 in Nolan (2009), a random variable $X$ is *stable* if, for any independent copies $X_1$ and $X_2$, and $a, b > 0$, we have $aX_1 + bX_2$ equal $cX + d$ in distribution, for some $c > 0$ and $d \in \mathbb{R}$.

[12] Every stable distribution has an *index of stability* $\alpha \in (0, 2]$. Nolan (2009) states that for a random variable $X$ with a stable distribution, $E[|X|^2] < \infty$ if and only if $\alpha = 2$ (p. 15). When $\alpha = 2$, the set of stable distributions is equivalent to the Gaussian family.

[13] For any continuous random variable $X$ and $Y$, $X \succeq_{disp} Y$ iff $X =_{st} \psi(Y)$ for some increasing $\psi$ which satisfies $\psi(y') - \psi(y) \geq y' - y$ whenever $y' \geq y$ by equation (3.B.13) in SS.

[14] Since $X/\alpha$ has mean $w$ and precision $\alpha^2/(1-\alpha^2)$, the posterior precision of $W$ is $1+\alpha^2/(1-\alpha^2) = 1/(1-\alpha^2)$ and its posterior mean is therefore $[(x/\alpha)\alpha^2/(1-\alpha^2)]/[1+\alpha^2/(1-\alpha^2)] = \alpha x$.

$$\ell(\alpha, u, c) \equiv \frac{u - \zeta(\alpha, c)}{\alpha}. \tag{15}$$

**Lemma 5 (Optionality)** *The search optionality value $\zeta(\alpha, c)$ falls in accuracy $\alpha$, when $\zeta(0, c) > -c = \zeta(1, c)$. Also, $\zeta(\alpha, c)/\sqrt{1-\alpha^2}$ monotonically falls in $\alpha$ to $-\infty$.*

The extreme cases are easy: The limit when accuracy vanishes $\alpha = 0$ is stationary search. Here, one searches if the best so far is less than the reservation value, i.e., $\zeta(0, c)$. With perfect accuracy $\alpha = 1$, noise and thus search optionality vanish. The model's richness arises from realistic intermediate accuracy levels $\alpha \in (0, 1)$.

A common ranking tool of search engines is the *click through rate* (CTR) — the chance the user clicks on ("explores") any web site after posting a query. This is our participation chance $\sigma_1$. By Theorems 1 and 2, for low outside option payoffs $u$, the CTR increases with a mean-preserving dispersion in the hidden factor and falls with a mean-preserving dispersion in the known factor. A similar tradeoff emerges for the search duration $\tau$. In our web search model, the known factor grows more dispersed and the hidden factor less so as accuracy rises. So the effect of greater accuracy on the CTR or search duration is unclear. Our first web search result resolves this doubt.

**Proposition 1 (Changing Accuracy)** *The quitting chance $q$ rises in accuracy $\alpha$ for low outside option payoffs $u < \zeta(\alpha, c)$, and otherwise falls in $\alpha$. The CTR and the search duration $\tau$ rise in accuracy $\alpha$ when $u$ is high enough, and fall for all lower $u$.*

For intuition, assume no outside option, so that the searcher never quits. As accuracy rises, the optionality value $\zeta(\alpha, c)$ of exploring inside options falls, and one stops sooner. But with a good outside option, the chance that $\alpha X_i > u - \zeta(\alpha, c)$ rises in accuracy: the searcher clicks through more, and the expected search time rises.

Since search engines constantly update their search algorithms, it is important to know whether a new algorithm enhances the user's payoff. By Proposition 1, the CTR, the expected search time, and the quitting chance are each non-monotone in search engine accuracy (Figure 4). As a result, using these statistics to measure search engine accuracy is unjustified. Since we show below that the searcher's welfare is increasing in accuracy, these statistics are also invalid welfare measures.

Proposition 1 crucially identifies a novel *potential conflict of interest between online shopping sites and consumers.* Suppose that the goal of a shopping search engine is to maximize the *sale chance* $1 - q$. This may be inconsistent with maximizing accuracy. For by Proposition 1, as accuracy rises, the quitting chance $q$ rises, and therefore the

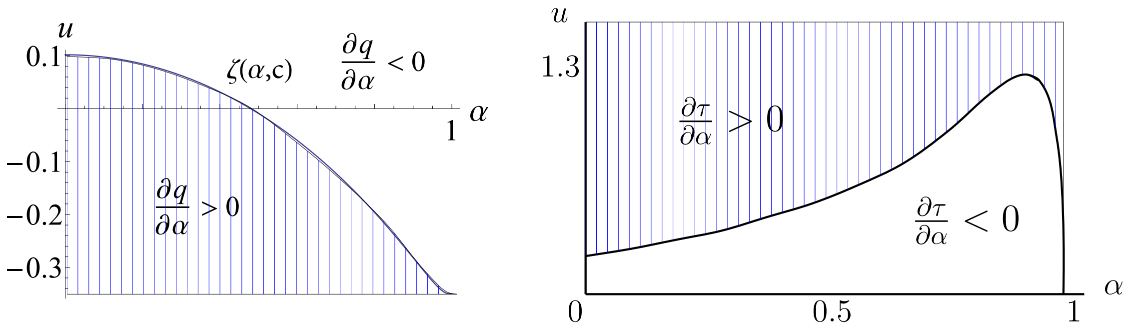


Figure 4: Both panels depict Proposition 1. At left, the slope of the quitting chance in $\alpha$ vanishes for the search optionality $\zeta(\alpha, c) = u$. At right, the expected search time falls/rises in $\alpha$ below/above the curve. The simulated graphs assume $c = 0.3, n = 6$.

sales chance falls, provided $u < \zeta(1, c)$ (Figure 4). Consequently,[15] when the price of the good is high (relative to the outside option), a shopping web site perversely earns higher profits from a less accurate search engine. Conversely, maximizing accuracy is sales-maximizing for low priced goods, since $q$ falls in $\alpha$ for outside options $u > \zeta(0, c)$.

## 8.2 The Value of a Search Engine

Let $\mathcal{V}(\alpha, c, u)$ be the (net) *pre-query value*, namely, the prospectively expected payoff of the user whose search engine has accuracy $\alpha$. This is the expectation of the initial value $V_0(u)$ in recursion (4), over possible known factors $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N$.

Let $\alpha X^*$ and $\sqrt{1-\alpha^2} Z^*$ be the random known and hidden factors for the consumed web site, assuming he exercises one. Then

$$\mathcal{V}(\alpha, c, u) = qu + (1-q)E[\alpha X^* + \sqrt{1-\alpha^2} Z^*] - \tau c.$$

By standard Envelope Theorem logic, two value derivatives are[16]

$$\frac{\partial \mathcal{V}(\alpha, c, u)}{\partial c} = -\tau \qquad \text{and} \qquad \frac{\partial \mathcal{V}(\alpha, c, u)}{\partial u} = q. \tag{16}$$

As noted after (12), the quitting chance $q$ increases in the outside option. So $\mathcal{V}_u(\alpha, c, u)$ increases in $u$, and $\mathcal{V}$ is strictly convex in $u$. Also, $\mathcal{V}$ is strictly convex in $c$.

Inspired by our Bayesian informational story in §8.1, we define the *value of the search engine* analogous to the value of information — namely, the gain over purely random search, i.e., $\Pi(\alpha, c, u) \equiv \mathcal{V}(\alpha, c, u) - \mathcal{V}(0, c, u)$. By Blackwell's Theorem,[17]

[15] For our applications here and in §8.3, the outside option $u$ is fixed, say proxied by the value of time. Then high and low priced goods correspond to high and low prices relative to $u$.

[16] By Lemma 2, $V_0(u)$ is differentiable in $u$ except at the cutoffs $\bar{w}_n = \mathcal{X}_n + \zeta(c)$. Since the distribution of known factors $\mathcal{X}_n$ is continuous, the pre-query value $\mathcal{V} = E[V_0(u)]$ is differentiable in $u$. By a similar argument, $\mathcal{V}$ is differentiable in $c$ (see §A in Online Appendix for details).

[17] Our dynamic strategy can be formulated at time zero as a set of contingent thresholds.

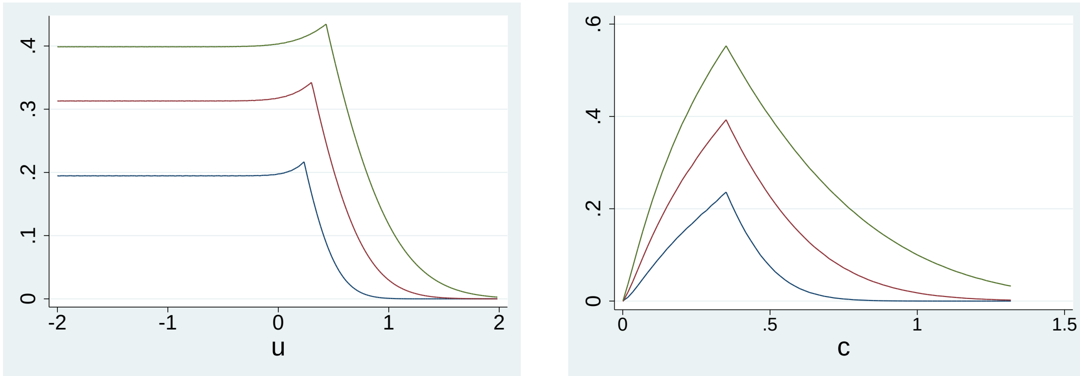


Figure 5: We plot, for accuracies $\alpha \in \{0.3, 0.5, 0.7\}$, how the (numerically-computed) search engine value is single-peaked in outside option $u$ (left) or clicking cost $c$ (right). The value is strictly positive as $u \to -\infty$ and vanishes as $u \to \infty$, or as $c \to \pm\infty$.

*the value of the search engine is monotone in* $\alpha$.[18] But intuitively, the user should strictly profit from a better search engine, or $\Pi(\alpha, c, u)$ should strictly rise in $\alpha$. Also:

**Proposition 2 (Changes in the Value)** $\Pi(\alpha, c, u)$ *is single-peaked in* $c > 0$ *and* $u$. *It vanishes as* $c \to 0$, $c \to \infty$, *or* $u \to \infty$, *and is boundedly positive as* $u \to -\infty$.

By Proposition 1, the search engine offers a positive gain to a web searcher with no outside option. Also, the value of a search engine is intuitively single-peaked in the $c$. For if $c = 0$, a user always clicks on all web sites and a search engine is useless. But if $c = \infty$, no web site is clicked, and a search engine is once again useless.

With no outside option ($u = -\infty$), a search engine reduces clicks and so helps. As $u$ increases, the search engine helps on intensive and extensive margins — it not only reduces the number of clicks, but also helps the user decide whether to click-through or not; therefore, the value of search engine initially is increasing in $u$. But for very large $u$, the user will not click on any web site, and the search engine is useless.

Search engine accuracy and *usability* $-c$ have much improved since 2000, and the number of web sites has exploded. Both trends are explained by two cross partials:

**Proposition 3 (Search Engine Synergies)** *Accuracy* $\alpha$ *and usability* $-c$ *are both complements to the number of web sites, namely* $\partial^2\mathcal{V}/\partial n\partial\alpha \geq 0$ *and* $\partial^2\mathcal{V}/\partial n\partial c \leq 0$.

These synergies have a feedback effect, for more accurate and useable search engine technology has surely encouraged the creation of more web sites.

[18] Given $\alpha_H$, one can create $\alpha_L < \alpha_H$, by adding zero mean Gaussian noise with variance $\alpha_H^2 - \alpha_L^2$ to the known factors, raising the hidden factor variance to $(1-\alpha_H^2) + (\alpha_H^2 - \alpha_L^2) = 1 - \alpha_L^2$.

## 8.3 Is Web Search Really Sequential?

Our model exhibits a known property of search and learning models, that encouraging search outcomes need not reduce search duration. Rosenfield and Shapiro (1981) showed that one need not even employ a cut-off strategy — for expectations rise after high draws. When $W_1$ is larger, so too are $\mathcal{X}_1, \ldots, \mathcal{X}_n$, and expected search duration rises. Our log-concavity assumptions help ensure the optimality of our threshold rule.

So inspired, we econometrically test our model. Intuitively, better earlier outcomes shorten the sequential search process. But this need not be so. Assume that search lasts $T \geq 1$ stages, and the first web site has payoff $W_1 = \mathcal{X}_1 + \mathcal{Z}_1$. Consider the OLS regression $T = \beta_0 + \beta_1 W_1 + \epsilon$ on data generated from our model. We claim that — fixing the CTR $\sigma_1 = 1 - G(u - \zeta(c))^n$ — the true coefficient obeys $\beta_1 > 0$, provided the outside option $u$ is large enough and search cost $c$ small enough. By Lemmas 1 and 3, the searcher clicks at stage $i$ if $\mathcal{X}_i + \zeta(c) > \Omega_i = \max(u, w_1, w_2, \ldots, w_i)$. In the limit $u \to \infty$ and $c \to 0$, and so $\zeta(c) \to \infty$, the stage $i$ search decision depends only on the known factor, clicking if $\mathcal{X}_i > u - \zeta(c) = \bar{\ell}$. As $W_1 = \mathcal{X}_1 + \mathcal{Z}_1$ is correlated with $\mathcal{X}_2$, one clicks the second web site more often with higher $W_1$ (see §D.7) — i.e. $\beta_1 > 0$.

In fact, search duration is not monotone in the first search outcome even ignoring its known factor. For consider the OLS regression $T = \beta_0 + \beta_2 \mathcal{Z}_1 + \epsilon$. The absolute true coefficient $|\beta_2|$ vanishes as $u \to \infty$ and $c \to 0$, fixing the CTR (see §D.7). This follows once more because the clicking decision depends on $\mathcal{X}_i$ but not $\mathcal{Z}_i$ for large $u$, very small $c$, but with $u - \zeta(c)$ fixed. So $T$ and $\mathcal{Z}_1$ are uncorrelated.

Recently, De Los Santos et al. (2012) (DHW) studies an online book market and test three *sine qua non* predictions of sequential search models. In their most relevant "test 3", DHW consider an OLS regression $T = \beta_0 + \beta_3 P_1 + \epsilon$ of the number of searches $T$ on the *price discount* $P_1$ at the first store. They assume that price discounts are learned after visiting the store,[19] and suggest that Weitzman's model requires $\beta_3 < 0$. For a higher first price discount intuitively leads the searcher to stop more often. Finding that $\beta_3$ is not statistically different from 0, DHW reject Weitzman's model.

But this logic misses selection effects. For a price learned *after* a store visit is best modeled as a hidden factor: $\mathcal{Z}_1 = P_1$. As our second regression shows, Weitzman's model yields a statistically insignificant coefficient $\beta_3$ on the hidden factor for a large outside option $u$ and search cost $c$ small relative to rewards — a plausible limit in

[19] DHW posit that consumer $i$'s payoff from buying at store $j$ is $u_{ij} = \delta_{ij} + \alpha_i p_j$. Consumer $i$ knows his gross utility $\delta_{ij}$ before searching store $j$, and learns the price discount $p_j$ after visiting store $j$. In our model, $\delta_{ij}$ is the known factor and the price discount $p_j$ is the hidden factor.

their context.[20] But if the searcher learns about the price discount $P_1$ *before* searching, then $P_1 = \mathcal{X}_1 + \mathcal{Z}_1$, where the searcher sees the known factor $\mathcal{X}_1$. In this case, our first regression shows that even $\beta_3 > 0$ is consistent with Weitzman's model when $u$ is large and $c$ is small. So really any sign of $\beta_3$ is consistent with Weitzman's model.

While DHW use data for cases *when users purchase from a web site* after searching, our regressions condition on participation. In §D.7, we study the regression $T = \beta_0 + \beta_3 P_1 + \epsilon$ given a final purchase. Venturing the extreme case when $P_1 = \mathcal{X}_1$, we show that if the hidden factor density has a thin tail, then $\beta_3 \geq 0$ as $u \to \infty$ and $c \to 0$, contrary to the DHW conjecture: Higher price discounts do not shorten search.

# A Prize Dispersion Proofs

## A.1 Hidden Dispersion and Duration: Proof of Theorem 1

Index the distribution so that $Z_t$ experiences a mean-enhancing dispersion as $t$ rises. Let $H_t$ and $\zeta_t(c)$ be the corresponding distribution function and search optionality.

**Claim A.1** *For any $\Delta \geq 0$, $H_t(\zeta_t(c) - \Delta)$ rises in the dispersion index $t$.*

Proof: Change variables in (5) to $z = H_t^{-1}(a) + \Delta$. Then

$$c = \int_{H_t(\zeta_t(c)-\Delta)}^{1} (1 - H_t(H_t^{-1}(a) + \Delta)) \frac{\partial H_t^{-1}(a)}{\partial a} da.$$

since $dz = [\partial H_t^{-1}(a)/\partial a]da$. Now, dispersion means that $\partial H_t^{-1}(a)/\partial a$ rises in $t$, i.e. the quantile function steepens. Also, $1 - H_t(H_t^{-1}(a) + \Delta)$ rises in $t$. To see why, put $s_t(a, \Delta) \equiv H_t(H_t^{-1}(a) + \Delta))$, so that $H_t^{-1}(s_t(a, \Delta)) - H_t^{-1}(a) \equiv \Delta$. Since $H_t^{-1}(s) - H_t^{-1}(a)$ rises in $t$ if $s > a$, equality $H_t^{-1}(s_t(a, \Delta)) - H_t^{-1}(a) = \Delta$ demands that $s_t(a, \Delta)$ fall in $t$. Since the integrand rises in $t$, the lower bound $H_t(\zeta_t(c) - \Delta)$ rises too. □

Let $t$ increase, changing $H_t$. By (8), the survival chance $\sigma_n$ rises in $\delta_t(\chi, c)$ and falls in $u - \zeta_t(c)$. Recalling (6), $\delta_t(\chi, c) = \int_0^\infty H_t(\zeta_t(c) - s)g(x + s)ds$ rises in $t$, since $H_t(\zeta_t(c) - s)$ rises, by Claim A.1. Thus, $\sigma_n$ increases if $u - \zeta_t(c)$ falls in $t$. Since a mean-enhancing dispersion in $\mathcal{Z}_t$ is a combination of a mean-preserving dispersion of

[20] For in DHW's data, about 5% of visits to online bookstores result in a transaction (15561 transactions from 327074 searches). Since DHW suggest less than 2 visits per search, the success chance is less than 10%; equivalently, the quitting chance is high, exceeding 90%. This in turn implies that consumers' outside option payoffs $u$ must be high relative to the size of the rewards, by (12). That consumers search despite such a low success chance implies a small search cost $c > 0$.

$\mathcal{Z}_t$ and a translation of $\mathcal{Z}_t$ — namely, adding a positive constant to $\mathcal{Z}_t$ — it suffices that $\zeta_t(c)$ rises in both cases. First, since $\zeta_t(c)$ rises in any MPS of $\mathcal{Z}_t$ by (5), $\zeta_t(c)$ rises as $\mathcal{Z}_t$ experiences a mean-preserving dispersion. Second, $\zeta_t(c)$ rises when $\mathcal{Z}_t$ experiences a translation by (5). Then $\zeta_t(c)$ rises in $t$, and thus $u - \zeta_t(c)$ falls. □

## A.2 Search Duration and Participation: Proof of Theorem 2

We next prove Theorem 2($a$) — for if the hidden factor remains unchanged, then so does $\zeta(c)$, and thus $\mathcal{X} + \zeta(c)$ stochastically rises/falls iff $\mathcal{X}$ stochastically rises/falls.

**Claim A.2** *Let $\mathcal{X}$ and $\mathcal{Z}$ change. If $\mathcal{X} + \zeta(c)$ stochastically rises (falls), then the participation chance $\sigma_1$ rises (falls). If $\mathcal{X} + \zeta(c)$ neither stochastically rises nor falls, but $\mathcal{X}$ grows more disperse, $\sigma_1$ falls for low outside options $u$, and rises for high $u$.*

Proof: If $\mathcal{X}_2 + \zeta_2(c) \succeq_{FSD} \mathcal{X}_1 + \zeta_1(c)$, then $G_2(u - \zeta_2(c)) \leq G_1(u - \zeta_1(c))$ for all $u$. The chance $\sigma_1 = 1 - G(u - \zeta(c))^N$ by (8) rises, and falls if $\mathcal{X} + \zeta(c)$ falls stochastically.

Next, $G_1(u - \zeta_1(c)) - G_2(u - \zeta_2(c))$ is single-crossing in $u$ if $\mathcal{X}_2 \succeq_{disp} \mathcal{X}_1$. For by (3.B.3) in SS, $\mathcal{X}_2 \succeq_{disp} \mathcal{X}_1$ iff $G_1(u) - G_2(u - k)$ is single crossing in $u$ for all $k$. If $\mathcal{X}_1 + \zeta_1(c)$ and $\mathcal{X}_2 + \zeta_2(c)$ are not stochastically ranked, then their cdfs intersect, by continuity — so that $P(\mathcal{X}_1 + \zeta_1(c) \leq \bar{u}) = P(\mathcal{X}_2 + \zeta_2(c) \leq \bar{u})$, for some $\bar{u}$. Thus, $G_1(u - \zeta_1(c)) - G_2(u - \zeta_2(c))$ is negative for $u < \bar{u}$ and positive for $u > \bar{u}$. □

To prove Theorem 2 ($b$), let $\vec{\Delta}_n = (\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_{n-1})$, where gaps are $\Delta_n \equiv x_n - x_{n+1} \geq 0$. As $\mathcal{X}_j - \mathcal{X}_n = \Sigma_{k=j}^{n-1}(\mathcal{X}_k - \mathcal{X}_{k+1}) = \Sigma_{k=j}^{n-1}\Delta_k$, the survival chance is

$$\sigma_n = P\left(\{\mathcal{X}_n + \zeta(c) \geq u\} \cap_{j<n} \left\{\zeta(c) - \Sigma_{k=j}^{n-1}\Delta_k \geq \mathcal{Z}_j\right\}\right)$$

If $\psi$ is the joint distribution of $\vec{\Delta}_n$ and $\mathcal{X}_n$, we may rewrite this as:

$$\sigma_n = \int_{x_n \in \mathbb{R}, \vec{\Delta}_n \in \mathbb{R}_+^{n-1}} \mathbb{I}_{\{x_n + \zeta(c) \geq u\}} \prod_{j=1}^{n-1} H(\zeta(c) - \Sigma_{k=j}^{n-1}\Delta_k) d\psi(\vec{\Delta}_n, x_n). \tag{17}$$

Let $G_n$ be the cdf of $\mathcal{X}_n$, where $\mathcal{X}_2 \geq_{disp} \mathcal{X}_1$. Since the gap between adjacent order statistics of $G_2$ exceeds that of $G_1$ for the dispersive order, the gaps between consecutive known factors $\vec{\Delta}_n$ jointly decrease stochastically as $\mathcal{X}$ grows less dispersive.

With no outside option ($u = -\infty$), the indicator $\mathbb{I}$ in (17) is one. As $\mathcal{X}$ grows less dispersive, $\vec{\Delta}_n \equiv \{\mathcal{X}_1 - \mathcal{X}_2, \ldots, \mathcal{X}_n - \mathcal{X}_{n+1}, \ldots, \mathcal{X}_{N-1} - \mathcal{X}_N\}$ falls stochastically, and so $\sigma_n$ rises. Next suppose $u > -\infty$. If $\mathcal{X}$ stochastically rises, then so does the order statistic $\mathcal{X}_n$ in (17), for $n = 1, \ldots, N$. Since $\mathbb{I}_{\{\mathcal{X}_n \geq u - \zeta(c)\}}$ rises in $\mathcal{X}_n$, so does $\sigma_n$. □

### A.3 Ex-ante Quitting Chance: Proof of Theorem 3

Recalling (11), let $q_n = \pi_n(u - \zeta(c), c)^N$ be the attraction for the hidden factor $\mathcal{Z}_n$, for $n = 1, 2$. If $\mathcal{Z}_2$ is a mean preserving dispersion of $\mathcal{Z}_1$, it suffices that $q_1 < q_2$ as $\mathcal{Z}$ iff $u$ is low enough. We claim $\pi_2(u - \zeta_2(c), c) \gtreqless \pi_1(u - \zeta_1(c), c)$ as $u \lesseqgtr \bar{u}$ for some $\bar{u}$.

Let $H_n$ be the cdf of $\mathcal{Z}_n$. Integrating (5) by parts, $\zeta_n(c) = -c + E[\mathcal{Z}] + \int_{-\infty}^{\zeta_n(c)} H_n(z)dz$. As dispersion implies a MPS, the integral $\int_{-\infty}^{a} H(z)dz$ rises, and so $\zeta_2(c) > \zeta_1(c)$.[21]

Let $\underline{H}_n$ be the cdf of $\min\{\mathcal{Z}_n, \zeta_n(c)\}$. Then $\underline{H}_2(z) - \underline{H}_1(z) = H_2(z) - H_1(z)$ obeys the reverse single crossing property if $z < \zeta_1(c)$, since $H_2^{-1}$ is steeper than $H_1^{-1}$. Likewise, $\underline{H}_2(z) - \underline{H}_1(z) = H_2(z) - 1 \le 0$ for $z \in [\zeta_1(c), \zeta_2(c))$, and $\underline{H}_2(z) - \underline{H}_1(z) = 0$ for $z > \zeta_2(c)$. Altogether, $\underline{H}_2 - \underline{H}_1$ has at most one sign change from + to −.

Because $\pi(u - \zeta(c), c) = P(\min(\mathcal{Z}, \zeta(c)) < u - \mathcal{X})$ from (12):

$$\pi(u - \zeta(c), c) = \int_{-\infty}^{\infty} P(\{\min(\mathcal{Z}, \zeta(c)) < s\} \cap \{s = u - \mathcal{X}\})ds = \int_{-\infty}^{\infty} \underline{H}_n(s) g(u - s)ds.$$

Now, $\pi_2(u - \zeta_2(c), c) - \pi_1(u - \zeta_1(c), c) = \int_{-\infty}^{\infty} [\underline{H}_2(s) - \underline{H}_1(s)] g(u - s)ds$ changes sign at most one sign from + to − as $u$ rises because this is true of $\underline{H}_2(s) - \underline{H}_1(s)$, and the density $g$ is log-concave, by Karlin and Rubin (1955).

Next, it is impossible that $\pi_1(u - \zeta_1(c), c) > \pi_2(u - \zeta_2(c), c)$ for all $u$. For if so, since $\pi_n(u - \zeta_n(c), c)$ is the cdf of $\mathcal{X} + \min\{\mathcal{Z}_n, \zeta_n(c)\}$, it follows that $\mathcal{X} + \min\{\mathcal{Z}_2, \zeta_2(c)\}$ strictly FSD dominates $\mathcal{X} + \min\{\mathcal{Z}_1, \zeta_1(c)\}$. This is impossible, as $E[\mathcal{X} + \min\{\mathcal{Z}_2, \zeta_2(c)\}] = E[\mathcal{X} + \min\{\mathcal{Z}_1, \zeta_1(c)\}]$, given $E[\mathcal{Z}_2] = E[\mathcal{Z}_1]$, and

$$E[\min\{\mathcal{Z}_n, \zeta_n(c)\}] - E[\mathcal{Z}_n] = \int_{\zeta_n(c)}^{\infty} (\zeta_n(c) - z)dH_n(z) = \int_{\zeta_n(c)}^{\infty} [1 - H_n(z)]dz = c.$$

by (5). Hence $\pi_2(u - \zeta_2(c), c) - \pi_1(u - \zeta_1(c), c)$ changes sign + to - once as $u$ rises. □

# B Selection Bias Proofs

## B.1 Stochastic Shifts of the Known Factor: Proof of Lemma 4

We analyze the delay chance $\delta(\chi, c)$ in Claim B.1 and then use it to prove Lemma 4.

**Claim B.1** *$\delta(\chi, c)$ falls in $c$ and is log-supermodular; also, $\delta(\chi, c)/G(\chi)$ falls in $\chi$.*

[21] When $\mathcal{Z}$ has full support, integration by parts requires $\lim_{z \to -\infty} zH(z) < \infty$. By l'Hopital's rule, $\lim_{z \to -\infty} zH(z) = \lim_{z \to -\infty} -z^2 h(z)$. This limit must vanish, for otherwise the second moment $\int_{-\infty}^{\infty} z^2 h(z)dz$ is infinite — impossible, as log-concave densities have finite moments (An, 1997).

Proof: Set $s = a - \chi$ in (6) so that $\delta(\chi, c) = \int_0^\infty H(\zeta(c) - s)\, g(s + \chi) ds$. Then $\zeta'(c) < 0$ implies $\delta_c(\chi, c) < 0$. Since $H(\zeta(c) - s)$ and $g(s + \chi)$ are log-supermodular in $(\zeta(c), s)$ and $(s, -\chi)$, resp., and partial integration preserves log-supermodularity (Karlin and Rinott, 1980), $\delta(\chi, c)$ is log-supermodular in $(\zeta(c), -\chi)$, and so in $(c, \chi)$.

Next,

$$\delta(\chi, c) = -H(\zeta(c))\, G(\chi) + \int_0^\infty h(\zeta(c) - s)\, G(s + \chi) ds. \tag{18}$$

Since $G(\chi)$ is log-concave, $G(s+\chi)/G(\chi)$ falls in $\chi$, and thus so does $\delta(\chi, c)/G(\chi)$. $\square$

Proof of Lemma 4: To infer the conditional distribution of $\mathcal{X}_n$ after the searcher enters stage $n$, let

$$\eta(\chi, n, c) = \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi) \tag{19}$$

so that $N\binom{N-1}{n-1}\eta(\chi, n, c)$ is the density of $\mathcal{X}_n$, by (8). Using (8) and (19),

$$P(\mathcal{X}_n \le a | \text{enters stage } n) = \frac{N\binom{N-1}{n-1} \int_{u-\zeta(c)}^a \eta(\chi, n, c) d\chi}{\sigma_n} = \frac{\int_{u-\zeta(c)}^a \eta(\chi, n, c) d\chi}{\int_{u-\zeta(c)}^\infty \eta(\chi, n, c) d\chi}. \tag{20}$$

This falls in $(N, c, u, -n)$ if its numerator is log-supermodular in $(a, N, c, u)$. By (19),

$$\frac{\partial}{\partial a} \log\left[\int_{u-\zeta(c)}^a \eta(\chi, n, c) d\chi\right] = \frac{\delta(a, c)^{n-1} G(a)^{N-n} g(a)}{\int_{u-\zeta(c)}^a \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi) d\chi}. \tag{21}$$

The RHS of (21) rises in $N$, since $G(a)/G(\chi) > 1$, for any $\chi < a$. Hence, the bracketed integral in (21) is log-supermodular in $(N, a)$, and thus (20) falls in $N$. Since the disappointment chance $\delta(\chi, c)$ is log-supermodular by Claim B.1, $\delta(a, c)/\delta(\chi, c)$ rises in $c$, if $\chi < a$. Since $u - \zeta(c)$ rises in $c$, the RHS of (21) rises in $c$. So $\int_{u-\zeta(c)}^a \eta(\chi, n, c) d\chi$ is log-supermodular in $(c, a)$. Then the conditional probability (20) falls in $c$. Similarly, the RHS of (21) rises in $u$ as $u - \zeta(c)$ rises in $u$, and so (20) falls in $u$. Finally, $\delta(\chi, c)/G(\chi)$ falls in $\chi$ by Claim B.1, and so $\delta(\chi, c)/G(\chi) > \delta(a, c)/G(a)$ for all $\chi < a$. Scaling (21) by $\delta(a, c)/G(a)$, we see that it falls in $n$, and so (20) rises in $n$. $\square$

## B.2 The Stopping Hazard Rate and Eventual Quitting Chance

We claimed after Lemma 4 that the stopping hazard rate $\mathcal{S}_n$ rises in the cost $c$. To prove this, let $E_{\mathcal{X}_n}$ be the stage-$n$ conditional expectation defined from (20). By (8):

$$1 - \mathcal{S}_n \equiv \frac{\sigma_{n+1}}{\sigma_n} = \frac{(N-n) \int_{u-\zeta(c)}^\infty [\delta(\chi, c)/G(\chi)] \eta(\chi, n, c) d\chi}{n \int_{u-\zeta(c)}^\infty \eta(\chi, n, c) d\chi} = \frac{N-n}{n} E_{\mathcal{X}_n}\left[\frac{\delta(\mathcal{X}_n, c)}{G(\mathcal{X}_n)}\right] \tag{22}$$

First, $\delta(\chi, c)/G(\chi)$ falls in $\chi$, by Claim B.1. The hazard rate $\mathcal{S}_n$ rises in $c$, as $\mathcal{X}_n$ stochastically rises in $c$, by Lemma 4, and $\delta(\chi, c)$ falls in $c$, by (6) and $\zeta'(c)<0$. $\square$

**Claim B.2** *For some threshold number of options $\bar{N} < \infty$, the eventual quitting chance $\bar{q}$ locally falls in search costs $c$ if $N \le \bar{N}$, and locally rises if $N > \bar{N}$.*

Proof: As $\bar{q} \equiv (q - q_0)/(1 - q_0)$, $q_0 = G(u - \zeta(c))^N$ by (10), and $q = \pi(u - \zeta(c), c)^N$ by (11), we have:

$$\bar{q} = \frac{\pi(u - \zeta(c), c)^N - G(u - \zeta(c))^N}{1 - G(u - \zeta(c))^N}. \tag{23}$$

Since $\pi(\chi, c) \equiv G(\chi) + \delta(\chi, c)$ by (7), when $N = 1$, by equation (6), we have

$$\bar{q} = \delta(u - \zeta(c), c)/[1 - G(u - \zeta(c)] = \int_{u-\zeta(c)}^{\infty} \frac{H(u - r)dG(r)}{1 - G(u - \zeta(c))} = E[H(u - R)]$$

where the known factor $R$ has cdf $G/[1 - G(u - \zeta(c))]$. Since $R$ is truncated with lower support $u - \zeta(c)$, and $\zeta' < 0$, $R$ rises stochastically in $c$. So $\partial\bar{q}/\partial c < 0$ if $N = 1$.

If $L_N(s) \equiv s^{N-1}/(1 - s^N)$, then $\partial \log(1 - s^N)/\partial s = -N L_N(s)$. By *sign equivalence* $\propto$:

$$\begin{aligned} \frac{\partial \bar{q}}{\partial c} &\propto -\frac{\partial}{\partial c} \log \left[ \frac{1 - \pi(u - \zeta(c), c)^N}{1 - G(u - \zeta(c))^N} \right] \\ &= N \frac{\partial \pi(u - \zeta(c), c)}{\partial c} L_N(\pi(u - \zeta(c), c)) + N \frac{\partial G(u - \zeta(c))}{\partial c} L_N(G(u - \zeta(c))) \end{aligned}$$

as $\zeta' < 0$. Since $\pi(u - \zeta(c), c) = G(u - \zeta(c)) + \int_{u-\zeta(c)}^{\infty} H(u - r)\, g(r) dr$ by (6)–(7), with derivative $-\zeta'(c) g(u - \zeta(c))[1 - H(\zeta(c))]$ in $c$, and $\partial G(u - \zeta(c))/\partial c = -\zeta'(c) g(u - \zeta(c))$, we have

$$\frac{\partial \bar{q}}{\partial c} \propto 1 - H(\zeta(c)) - \frac{L_N(G(u - \zeta(c))}{L_N(\pi(u - \zeta(c), c))}. \tag{24}$$

The last term in (24) vanishes as $N \to \infty$, given $0 < G(u - \zeta(c), c) < \pi(u - \zeta(c), c) < 1$ by (7), and $L_N(s) \approx s^N$ for $0 < s < 1$ and large $N$. So $\partial\bar{q}/\partial c > 0$ for all large $N$. Now, $\partial\bar{q}/\partial c$ single-crosses 0 in $N$, since

$$\frac{\partial^2(\log L_N(s))}{\partial s \partial N} = \frac{N \log(s) s^N + 1 - s^N}{s(1 - s^N)^2} \propto s^N \log(s^N) + 1 - s^N \equiv RHS.$$

Also, $RHS = 0$ when $s = 1$, and falls in $s^N$ for $s < 1$. Since $RHS > 0$ for $s < 1$, $L_N(s)$ is log-supermodular in $(N, s)$. Since $G(\chi) < \pi(\chi)$ for all $\chi$ by (7), the (signed) last term in (24) rises in $N$. Hence, $\partial\bar{q}/\partial c \lesseqgtr 0$ as $N \lesseqgtr \bar{N}$, for some $\bar{N} > 1$. $\square$

## B.3 Hazard Rates: Proof of Theorem 4

A. Quitting Hazard Rates. Given Claim B.3, for low search costs $c$, the quitting hazard rate $\mathcal{Q}_n \equiv q_n/\sigma_n$ rises in $n$ since the survival chance $\sigma_n$ falls and $q_n$ rises in $n$.

**Claim B.3** *The stage n quitting chance $q_n$ in* (10) *rises in n for small c, hump-shaped in n for middle c, and falls in n for large c.*

Proof: By (10), $q_{n+1}/q_n = [(N-n)/(n+1)]\delta(u-\zeta(c),c)/G(u-\zeta(c))$ falls in $n$. If $\delta(u-\zeta(c),c)/G(u-\zeta(c)) < 1/N$, then $q_{n+1}/q_n < 1$ for all $n$, and hence $q_n$ falls in $n$. Similarly, if $\delta(u-\zeta(c),c)/G(u-\zeta(c)) > N$, then $q_n/q_{n-1} > 1$ for all $n$ and thus $q_n$ rises in $n$. If $1/N < \delta(u-\zeta(c),c)/G(u-\zeta(c)) < N$, then $q_n$ rises and then falls in $n$.

To see how the threshold $\delta(u-\zeta(c),c)/G(u-\zeta(c))$ changes in $c$, use (6) to derive

$$\frac{\delta(u-\zeta(c),c)}{G(u-\zeta(c))} = \frac{1}{G(u-\zeta(c))}\int_0^\infty H(\zeta(c)-r)g(r+u-\zeta(c))dr. \tag{25}$$

Since $G$ is log-concave and $\zeta'(c) < 0$, the next product falls in $c$, and vanishes as $c \to \infty$, since then $\zeta(c) \to -\infty$ — and the ratio (25) falls in $c$ and vanishes as $c \to \infty$:

$$\frac{g(r+u-\zeta(c))}{G(u-\zeta(c))} = \frac{g(r+u-\zeta(c)}{G(r+u-\zeta(c))}\frac{G(r+u-\zeta(c))}{G(u-\zeta(c))}.$$

Finally, we argue (25) explodes as $c \to 0$: since $\zeta(c) \to \infty$ as $c \to 0$, $H(\zeta(c)-r) \to 1$ and $\lim_{c\to 0}\int_0^\infty g(s+u-\zeta(c))ds/G(u-\zeta(c)) = \lim_{\zeta(c)\to\infty}[1-G(u-\zeta(c))]/G(u-\zeta(c)) = \infty$. Then $\delta(u-\zeta(c),c)/G(u-\zeta(c))$ falls from $\infty$ to 0 as $c$ rises from 0 to $\infty$. □

B. Exercising Hazard Rates. Equation (26) below provides a formula for the hazard rate of exercising an inside option $\mathcal{E}_n$. It rises in $n$ as $\mathcal{X}_n$ falls stochastically in $n$ by Lemma 4 and the integral (26) falls in $\mathcal{X}_n$, by log-concavity of $g$.

**Claim B.4** *We have the formula:*

$$\mathcal{E}_n = 1 - H\left(\zeta(c)\right) + E_{\mathcal{X}_n}\left[\int_0^\infty h\left(\zeta(c)-s\right)g(s+\mathcal{X}_n)/g(\mathcal{X}_n)ds\right]. \tag{26}$$

Proof: Formula (8) yields $\sigma_{n+1} = \binom{N}{n}\int_{u-\zeta(c)}^\infty \delta(\chi,c)^n dG(\chi)^{N-n}$. Integrating by parts,

$$\begin{aligned}\sigma_{n+1} &= \binom{N}{n}\left[-\delta(u-\zeta(c),c)^n G(u-\zeta(c))^{N-n} - n\int_{u-\zeta(c)}^\infty G(\chi)^{N-n}\delta(\chi,c)^{n-1}\delta_\chi(\chi,c)d\chi\right] \\ &= -q_n - \binom{N}{n}n\int_{u-\zeta(c)}^\infty G(\chi)^{N-n}\delta(\chi,c)^{n-1}\delta_\chi(\chi,c)d\chi. \end{aligned} \tag{27}$$

using $\delta(\infty, c) = 0$ by (6) and the $q_n$ formula in (10). Differentiating (18), $\delta_\chi(\chi, c) = -H(\zeta(c))\, g(\chi) + \int_0^\infty h(\zeta(c) - s)\, g(s + \chi) ds$. Substitute $\delta_\chi$ into (27), and divide by $\sigma_n$:

$$\frac{\sigma_{n+1}}{\sigma_n} = -\frac{q_n}{\sigma_n} + H(\zeta(c)) - \frac{\int_{u-\zeta(c)}^\infty \left(\int_0^\infty h(\zeta(c) - s)\,[g(s+\chi)/g(\chi)] ds\right) \eta(\chi, n, c) d\chi}{\int_{u-\zeta(c)}^\infty \eta(\chi, n, c) d\chi}.$$

using $\eta(\chi, n, c) = \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi)$ from (19). Substituting this and $\mathcal{Q}_n \equiv q_n/\sigma_n$ into $\mathcal{E}_n \equiv \mathcal{S}_n - \mathcal{Q}_n = 1 - \sigma_{n+1}/\sigma_n - \mathcal{Q}_n$ gives formula (26). $\square$

C. Recall Hazard Rates. We characterize $\mathcal{R}_n = \mathcal{E}_n - \mathcal{K}_n$ using (26) and a new formula for $\mathcal{K}_n$. Let $\bar{\mathcal{K}}_n(\chi_n, c)$ be the interim striking hazard rate on entering stage $n$ if $\mathcal{X}_n = \chi_n$. For the $E_{\mathcal{X}_n}$ operator (§B.2), *the striking hazard rate is* $\mathcal{K}_n = E_{\mathcal{X}_n}[\bar{\mathcal{K}}_n(\mathcal{X}_n, c)]$. Consider the chance of striking in stage $n$ if $\mathcal{X}_n = x_n$ and $\mathcal{Z}_n < \zeta(c)$:

$$A(\chi_n, c) \equiv \int_{u-\chi_n}^{\zeta(c)} h(z_n) \left[\frac{\int_{\chi_n}^\infty H(\chi_n + z_n - r) g(r) dr}{\int_{\chi_n}^\infty H(\chi_n + \zeta(c) - r) g(r) dr}\right]^{n-1} \left[\frac{G(\chi_n + z_n - \zeta(c))}{G(\chi_n)}\right]^{N-n} dz_n. \quad (28)$$

**Claim B.5** *The interim striking hazard rate is* $\bar{\mathcal{K}}_n(\chi_n, c) = 1 - H(\zeta(c)) + A(\chi_n, c)$.

Proof: First, assume the searcher strikes option $n$ if he explores it. Assume $\mathcal{X}_n = \chi_n$. After arriving at stage $n$, he strikes option $n$ iff its payoff ($a$) dominates all prior inside options, namely $\chi_n + \mathcal{Z}_n > \mathcal{X}_j + \mathcal{Z}_j$ for all $j < n$, ($b$) dominates the outside option $\chi_n + \mathcal{Z}_n > u$, and ($c$) *ex ante* dominates option $n+1$, namely $\chi_n + \mathcal{Z}_n > \mathcal{X}_{n+1} + \zeta(c)$.

Assume $\mathcal{Z}_n \geq \zeta(c)$ and the searcher enters stage $n$. First, ($a$) is satisfied because $\chi_n + \mathcal{Z}_n \geq \chi_n + \zeta(c) > \mathcal{X}_j + \mathcal{Z}_j$ for all $j < n$ — for if $\chi_n + \zeta(c) \leq \mathcal{X}_j + \mathcal{Z}_j$, then the searcher never explores option $n$. Second, ($b$) is satisfied because $\chi_n + \mathcal{Z}_n \geq \chi_n + \zeta(c) > u$ — for if $\chi_n + \zeta(c) \leq u$, then the searcher quits rather than explores option $n$. Finally, ($c$) is met since $\mathcal{X}_{n+1} < \chi_n$ and $\mathcal{Z}_n \geq \zeta(c)$. In sum, the searcher strikes option $n$ with chance $P(\mathcal{Z}_n \geq \zeta(c)) = 1 - H(\zeta(c))$ if he enters stage $n$.

Next, assume $\mathcal{Z}_n < \zeta(c)$. Given $\mathcal{X}_n = \chi_n$, by the Markov property of order statistics, the joint distribution of earlier inside options $(\mathcal{X}_j, \mathcal{Z}_j)$ for all $j < n$ is the same as that of $(n-1)$ *i.i.d* random options with a known factor $\mathcal{X} > \chi_n$ and a payoff $\mathcal{X} + \mathcal{Z} < \chi_n + \zeta(c)$. So the probability that each prior option satisfies condition ($a$) is

$$P(\mathcal{X} + \mathcal{Z} < \chi_n + \mathcal{Z}_n | \mathcal{X} > \chi_n, \mathcal{X} + \mathcal{Z} < \chi_n + \zeta(c)) = \frac{\int_{\chi_n}^\infty H(\chi_n + \mathcal{Z}_n - r) g(r) dr}{\int_{\chi_n}^\infty H(\chi_n + \zeta(c) - r) g(r) dr}.$$

By the Markov property again, $\mathcal{X}_{n+1}$ is the first order statistic of $(N - n)$ i.i.d.r.v.s with cdf $G/G(\chi_n)$ on $(-\infty, \chi_n]$. Event ($c$) has chance $[G(\chi_n + \mathcal{Z}_n - \zeta(c))/G(\chi_n)]^{N-n}$.

As event $(b)$ holds if $\mathcal{Z}_n > u - \chi_n$, starting stage $n$, the chance of $(a)$–$(c)$ is $A(\chi_n, c)$. $\square$

**Claim B.6 (Striking Hazard Rate Formula)** $\mathcal{K}_n = E_{\mathcal{X}_n}[\bar{\mathcal{K}}_n(\mathcal{X}_n, c)]$ *equals:*

$$\mathcal{K}_n = 1 - H(\zeta(c)) + E_{\mathcal{X}_n}\left( \int_0^\infty h\left(\zeta(c) - s\right) \left[ \frac{\int_s^\infty H(\zeta(c)-t) g(t+\mathcal{X}_n) dt}{\int_0^\infty H(\zeta(c)-t) g(t+\mathcal{X}_n) dt} \right]^{n-1} \frac{g(s+\mathcal{X}_n)}{g(\mathcal{X}_n)} ds \right)$$

Proof: Recall that the expectation $E_{\mathcal{X}_n}$ uses density $\eta(\chi_n, n, c)$ in (20). Given (28) and $\delta(\chi_n, c) = \int_{\chi_n}^\infty H(\chi_n + \zeta(c) - r) g(r) dr$ by (6), the expectation $E_{\mathcal{X}_n}[A(\mathcal{X}_n, c)]$ equals

$$\frac{\int_{u-\zeta(c)}^\infty \left[ \int_{u-\chi_n}^{\zeta(c)} h(z_n) \left[ \frac{\int_{\chi_n}^\infty H(\chi_n + z_n - r) g(r) dr}{\delta(\chi_n, c)} \right]^{n-1} \left[ \frac{G(\chi_n + z_n - \zeta(c))}{G(\chi_n)} \right]^{N-n} dz_n \right] \eta(\chi_n, n, c) d\chi_n}{\int_{u-\zeta(c)}^\infty \eta(\chi, n, c) d\chi}.$$

Substitute $\eta(\chi_n, n, c) = \delta(\chi_n, c)^{n-1} G(\chi_n)^{N-n} g(\chi_n)$ from (19), and change variables in the numerator: $\chi = \chi_n + z_n - \zeta(c)$, $s = \zeta(c) - z_n$ and $t = r - \chi$ with supports $\chi \in [u - \zeta(c), \infty)$, $s \in [0, \infty)$ and $t \in [s, \infty)$. Then $E_{\mathcal{X}_n}[A(\mathcal{X}_n, c)]$ becomes

$$\frac{\int_{u-\zeta(c)}^\infty \int_0^\infty h\left(\zeta(c) - s\right) \left[ \int_s^\infty H(\zeta(c) - t) g(t + \chi) dt \right]^{n-1} G(\chi)^{N-n} g(s + \chi) ds d\chi}{\int_{u-\zeta(c)}^\infty \eta(\chi, n, c) d\chi}.$$

Since $\eta(\chi, n, c) = [\int_0^\infty H(\zeta(c) - t) g(t + \chi) dt]^{n-1} G(\chi)^{N-n} g(\chi)$ by (6) and (19), and recalling that $E_{\mathcal{X}_n}$ is defined using the probability density $\eta(\chi_n, n, c)$, we have

$$E_{\mathcal{X}_n}[A(\mathcal{X}_n, c)] = E_{\mathcal{X}_n}\left[ \int_0^\infty h\left(\zeta(c) - s\right) \frac{g(s+\mathcal{X}_n)}{g(\mathcal{X}_n)} \left[ \frac{\int_s^\infty H(\zeta(c)-t) g(t+\mathcal{X}_n) dt}{\int_0^\infty H(\zeta(c)-t) g(t+\mathcal{X}_n) dt} \right]^{n-1} ds \right].$$

By Claims B.4 and B.6, the recall hazard rate is $\mathcal{R}_n = \mathcal{E}_n - \mathcal{K}_n = E_{\mathcal{X}_n}[B(\mathcal{X}_n, n)]$, where

$$B(\chi, n) \equiv \int_0^\infty h\left(\zeta(c) - s\right) \frac{g(s+\chi)}{g(\chi)} \left( 1 - \left[ \frac{\int_s^\infty H(\zeta(c)-t) g(t+\chi) dt}{\int_0^\infty H(\zeta(c)-t) g(t+\chi) dt} \right]^{n-1} \right) ds. \quad (29)$$

Here, $B(\chi, n) g(\chi)$ is the recall chance at stage $n$ if one exercises an inside option with payoff $\chi + \zeta(c)$ immediately. For $h(\zeta(c) - s) g(s + \chi)$ is the probability density of an option with known factor $\chi + s$ and payoff $\chi + \zeta(c)$. The last term is the chance that one other option has a known factor below $\chi + s$, assuming its payoff is below $\chi + \zeta(c)$.

**Claim B.7** *The function $B(\chi, n)$ weakly falls in $\chi$.*

Proof: Put $\nu(s, \chi, \zeta(c)) \equiv \int_s^\infty H(\zeta(c) - t) g(t + \chi) dt$. Then $\nu(0, \chi, \zeta(c)) = \delta(\chi, c)$ by (6).

Rework (29), integrate by parts, and simplify via $\nu_s(s,\chi,\zeta(c)) = -H(\zeta(c)-s)g(s+\chi)$:

$$\begin{aligned} B(\chi,n) &= -\int_0^\infty \left(1-\left[\frac{\nu(s,\chi,\zeta(c))}{\nu(0,\chi,\zeta(c))}\right]^{n-1}\right) d\left[\int_s^\infty h\left(\zeta(c)-t\right)\frac{g(t+\chi)}{g(\chi)}dt\right] \\ &= -\int_0^\infty \left[\int_s^\infty h\left(\zeta(c)-t\right)\frac{g(t+\chi)}{g(\chi)}dt\right] d\left[\frac{\nu(s,\chi,\zeta(c))}{\nu(0,\chi,\zeta(c))}\right]^{n-1} \end{aligned} \tag{30}$$

as $(\partial/\partial s)\int_s^\infty h\left(\zeta(c)-t\right)[g(t+\chi)/g(\chi)]dt = -h\left(\zeta(c)-s\right)g(s+\chi)/g(\chi)$. So (30) gives:

$$\frac{B(\chi,n)}{n-1} = \int_0^\infty \frac{H(\zeta(c)-s)g(s+\chi)}{g(\chi)}\left[\frac{\int_s^\infty h\left(\zeta(c)-t\right)g(t+\chi)dt}{\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt}\right]\left[\frac{\nu(s,\chi,\zeta(c))}{\nu(0,\chi,\zeta(c))}\right]^{n-1} ds.$$

First, $g(s+\chi)/g(\chi)$ falls in $\chi$. Next, since $H$ and $g$ are log-concave, $H(\zeta(c)-t)g(t+\chi)$ is log-supermodular in $(\zeta(c),-\chi,t)$, as is the integral $\nu(s,\chi,\zeta(c))$ in $(\zeta(c),-\chi)$, by Karlin and Rinott (1980). So $\nu(s,\chi,\zeta(c))$ is log-submodular in $(\zeta(c),\chi)$. The first bracketed term $\partial\log[\nu(s,\chi,\zeta(c))]/\partial\zeta(c)$ falls in $\chi$. The last term likewise falls in $\chi$. □

Proof of Theorem 4: The hazard rate $\mathcal{R}_n = \mathcal{E}_n - \mathcal{K}_n$ rises in $n$ by (29), as $B(\chi,n)$ rises in $n$ and falls in $\chi$ (Claim B.7), and $\mathcal{X}_n$ falls stochastically in $n$ by Lemma 4. □

# C Stationary Benchmark Proofs

## C.1 Equivalent Thin Tail Characterizations

Let $\ell = \lim_{\chi\to F^{-1}(1)} f(\chi)/[1-F(\chi)]$. If a cdf $F$ has a thin tail, then $\ell=\infty$.

**Claim C.1** *If $f'$ exists and $F^{-1}(1)=\infty$, then $\lim_{\chi\to F^{-1}(1)} f(s+\chi)/f(\chi) = e^{-s\ell}, \forall s>0$.*

Proof: As $F^{-1}(1)=\infty$, $\lim_{\gamma\to 1} f(F^{-1}(\gamma))=0$. Now, l'Hôpital's rule implies that $\ell = \lim_{\chi\to F^{-1}(1)} f(\chi)/[1-F(\chi)] = \lim_{\chi\to F^{-1}(1)} -f'(\chi)/f(\chi)$. Then for all $s>0$:

$$\lim_{\chi\to\infty}\log\left(\frac{f(s+\chi)}{f(\chi)}\right) = \lim_{\chi\to\infty}\int_0^s \frac{f'(r+\chi)}{f(r+\chi)}dr = \int_0^s \lim_{\chi\to\infty}\frac{f'(r+\chi)}{f(r+\chi)}dr = -s\ell,$$

by the Monotone Convergence Theorem, for $f'/f$ is monotone if $f$ is log-concave. So

$$\lim_{\chi\to\infty}\frac{f(s+\chi)}{f(\chi)} = \lim_{\chi\to\infty}\exp\left[\log\left(\frac{f(s+\chi)}{f(\chi)}\right)\right] = \exp\left[\lim_{\chi\to\infty}\log\left(\frac{f(s+\chi)}{f(\chi)}\right)\right] = e^{-s\ell}\ \ \forall s>0.$$

**Claim C.2** *If $F''=f'$ exists and $F^{-1}(1)=\infty$, then $F$ has a thin tail if and only if $\lim_{\chi\to F^{-1}(1)} f(s+\chi)/f(\chi)=0, \forall s>0$.*

PROOF: ($\Rightarrow$) Given a thin tail, $\ell = \infty$ and $f(s{+}\chi)/f(\chi) \to 0$ for $s > 0$ by Lemma C.1.

($\Leftarrow$) If $\lim_{\chi\to\infty} f(s+\chi)/f(\chi) = 0\ \forall s > 0$, then $\lim_{\chi\to F^{-1}(1)} f(\chi)/[1-F(\chi)]$ equals

$$\lim_{\chi\to\infty}\left(\int_0^\infty \frac{f(s+\chi)}{f(\chi)}ds\right)^{-1} = \left(\lim_{\chi\to\infty}\int_0^\infty \frac{f(s+\chi)}{f(\chi)}ds\right)^{-1} = \left(\int_0^\infty \lim_{\chi\to\infty}\frac{f(s+\chi)}{f(\chi)}ds\right)^{-1} = \infty.$$

by continuity and the Monotone Convergence Theorem. Hence, $F$ has a thin tail. □

## C.2 Number of Options: Proof of Theorem 6

Index the striking, recall and quitting hazard rates by the number of options $N$.

**Claim C.3** *The striking hazard rate $\mathcal{K}_n^N$ falls in the total number of options $N$.*

PROOF: As $g(s+\chi)/g(\chi)$ falls in $\chi$ by log-concavity, and $\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt$ is log-submodular in $(s,\chi)$ by log-concavity of $H$ and $g$, Claim B.6's bracketed integral falls in $\mathcal{X}_n$. As $\mathcal{X}_n$ stochastically rises in $N$ (Lemma 4), $\mathcal{K}_n^N$ falls in $N$. □

As the $N$ option striking hazard rate $\mathcal{K}_n^N$ falls in $N$, $\mathcal{K}_n^\infty$ exists.

**Claim C.4** $\mathcal{K}_n^\infty = 1 - H(\zeta(c))$ *if $G$ has a thin tail, and* $\mathcal{K}_n^\infty > 1 - H(\zeta(c))$ *otherwise.*

PROOF: In Claim B.6, the $n^{th}$ known factor $\mathcal{X}_n \to G^{-1}(1) = \infty$ in probability, as $N \to \infty$. If $G$ has a thin tail, then $\lim_{\chi\to\infty} g(s+\chi)/g(\chi) = 0$ for $s>0$, by Claim C.2, and so $g(s{+}\mathcal{X}_n)/g(\mathcal{X}_n)$ vanishes as $N \to \infty$. By Claim B.6, $\lim_{N\to\infty}\mathcal{K}_n^N = 1-H(\zeta(c))$.

Assume $G$ lacks a thin tail. Let $\Gamma(s,\mathcal{X}_n)$ be the bracketed term in Claim B.6's integral for $\mathcal{K}_n$. As $\lim_{\chi\to\infty} g(t+\chi)/g(\chi) = e^{-t\ell}$ for $\ell \in (0,\infty)$ by Claim C.1,

$$\Gamma(s,\chi) \equiv \frac{\int_s^\infty H(\zeta(c)-t)g(t+\chi)/g(\chi)dt}{\int_0^\infty H(\zeta(c)-t)g(t+\chi)/g(\chi)dt} \to \frac{\int_s^\infty H(\zeta(c)-t)e^{-\ell t}dt}{\int_0^\infty H(\zeta(c)-t)e^{-\ell t}dt} > 0 \text{ as } \chi \to \infty. \tag{31}$$

As $\lim_{\chi\to\infty} g(s+\chi)/g(\chi) = e^{-s\ell} > 0$ and $\mathcal{X}_n \to G^{-1}(1) = \infty$ in probability as $N \uparrow \infty$, the second term in the Claim B.6 formula is positive as $N\uparrow\infty$: $\mathcal{K}_n^\infty > 1 - H(\zeta(c))$. □

**Claim C.5** $\mathcal{R}_n^N$ *falls in the number of options $N$ and* $\mathcal{R}_n^\infty = 0$ *iff $G$ has a thin tail.*

PROOF: As $B(\chi,n)$ falls in $\chi$ by Claim B.7, $\mathcal{R}_n^N = E_{\mathcal{X}_n}[B(\mathcal{X}_n,n)]$ falls in $N$ because $\mathcal{X}_n$ rises stochastically in $N$ by Lemma 4. Since $\mathcal{R}_n^N \geq 0$ falls in $N$, $\lim_{N\to\infty}\mathcal{R}_n^N = \lim_{N\to\infty} E_{\mathcal{X}_n}[B(\mathcal{X}_n,n)]$ exists. Then $\lim_{N\to\infty}\mathcal{R}_n^N = \lim_{\chi\to G^{-1}(1)} B(\chi,n)$, because $\mathcal{X}_n$ converges to $G^{-1}(1)$ in probability as $N \to \infty$.

If $G$ has a thin tail, then $g(s+\chi)/g(\chi)$ vanishes for all $s>0$ as $\chi \to G^{-1}(1)=\infty$ by Claim C.2, and so $\lim_{\chi \to G^{-1}(1)} B(\chi,n)=0$ by (29).

If $G$ does not have a thin tail, then the parenthesized term in (29) is $1-\Gamma(s,\chi)^{n-1}$ (recalling (31)), and its limit is boundedly positive as $\chi \to G^{-1}(1)$, for any $s>0$. Also, $\lim_{\chi \to G^{-1}(1)} g(s+\chi)/g(\chi)=e^{-s\ell}$ for some $\ell \in (0,\infty)$ by Claim C.1, and thus it is boundedly positive for any $s>0$. Hence, $\lim_{\chi \to G^{-1}(1)} B(\chi,n)>0$, by (29).

Altogether, $\lim_{N\to\infty} \mathcal{R}_n^N = \lim_{\chi \to G^{-1}(1)} B(\chi,n)=0$ iff $G$ has a thin tail. □

**Claim C.6** *$\mathcal{Q}_n^N$ falls in the total number of options $N$ and has limit $\mathcal{Q}_n^\infty=0$.*

Proof: Expanding $\mathcal{Q}_n \equiv q_n/\sigma_n$ using (8) and (10):

$$\mathcal{Q}_n^N = \frac{\delta(u-\zeta(c),c)^n G(u-\zeta(c))^{N-n}}{n\int_{u-\zeta(c)}^{\infty} \delta(\chi,c)^{n-1} G(\chi)^{N-n} g(\chi) d\chi}.$$

Easily, $\mathcal{Q}_n^N$ falls in $N$, since $G(\chi)/G(u-\zeta(c))>1$ except at $\chi=u-\zeta(c)$, and so $[G(\chi)/G(u-\zeta(c))]^{N-n}$ is monotone in $N$. By the monotone convergence theorem, we can swap the (infinite) limit as $N\to\infty$ and integration: $\lim\limits_{N\to\infty} \mathcal{Q}_n^N = \mathcal{Q}_n^\infty = 0$. □

# D Web Search Proofs

## D.1 Optionality in a Gaussian World: Proof of Lemma 5

As $\alpha$ increases, the hidden factor experiences a mean preserving contraction, and so $\zeta(\alpha,c)$ falls. As $\alpha \uparrow 1$, (14) reduces to $c=\int_{\zeta(\alpha,c)}^{0} ds = -\zeta(\alpha,c)$, and so $\zeta(\alpha,c)\downarrow -c$.

Next, change variables to $s'=s/\sqrt{1-\alpha^2}$ in (14), and let $z(\alpha)\equiv \zeta(\alpha,c)/\sqrt{1-\alpha^2}$. This yields $c/\sqrt{1-\alpha^2} = \int_{z(\alpha)}^{\infty} [1-\Phi(s')]\, ds'$. The LHS rises to $\infty$ as $\alpha$ rises to 1. Since the mean of a left truncated Gaussian distribution is finite, $\int_{z(\alpha)}^{\infty} [1-\Phi(s')]\, ds' = E[\max\{z(\alpha),S'\}]-z(\alpha)$ is finite if $z(\alpha)>-\infty$. So $z(\alpha)\downarrow -\infty$ as $\alpha$ rises to 1. □

**Claim D.1** *The search optionality $\zeta(\alpha,c)$ is concave-convex in the accuracy $\alpha$.*

Proof: Again, let $z(\alpha)=\zeta(\alpha,c)/\sqrt{1-\alpha^2}$. Differentiation yields

$$\begin{aligned}
\zeta_\alpha(\alpha,c) &= -\frac{\alpha}{\sqrt{1-\alpha^2}} \frac{\phi(z(\alpha))}{1-\Phi(z(\alpha))} < 0 \\
\zeta_{\alpha\alpha}(\alpha,c) &= \frac{-1}{(\sqrt{1-\alpha^2})^3} \frac{\phi(z(\alpha))}{1-\Phi(z(\alpha))} \left(1-\alpha^2\left\{\frac{\phi(z(\alpha))}{1-\Phi(z(\alpha))}-z(\alpha)\right\}^2\right). \quad (32)
\end{aligned}$$

First, $\phi(z)/[1-\Phi(z)]-z\to\infty$ as $z\downarrow -\infty$, and is positive and decreasing. For it is well-known that the conditional mean $y(z)=E[Z|Z>z]\equiv \int_z^\infty s d\Phi(s)/[1-\Phi(z)]$

equals the inverse Mill's ratio, namely, $\phi(z)/[1-\Phi(z)]$. But $y(z)-z$ has slope less than one, by log-concavity of the Gaussian. Since $z(\alpha)$ decreases to $-\infty$ as $\alpha \uparrow 1$, the bracketed term in (32) reverse single-crosses through 0 as $\alpha$ traverses $[0,1]$. $\square$

## D.2 Accuracy and Behavior: Proof of Proposition 1

Set $\mathcal{X}=\alpha X$ and $\mathcal{Z}=\sqrt{1-\alpha^2}Z$ in (7) to derive the attraction

$$\pi(\alpha,x)=\int_0^\infty \Phi\left(\frac{\zeta(\alpha,c)-\alpha s}{\sqrt{1-\alpha^2}}\right)\phi(x+s)ds+\Phi(x). \tag{33}$$

By (12), $q=\pi(\alpha,\ell(\alpha,u,c))^N$. We argue in Appendix §D.3 that

$$\frac{\partial\pi(\alpha,\ell(\alpha,u,c))}{\partial\alpha}=-\left[1-\Phi\left(\zeta(\alpha,c)/\sqrt{1-\alpha^2}\right)\right]\ell(\alpha,u,c)\phi(\ell(\alpha,u,c))/\alpha. \tag{34}$$

We have $\partial q/\partial\alpha>0$ iff $\partial\pi(\alpha,\ell(\alpha,u,c))/\partial\alpha>0$, and so by (34), iff $\ell(\alpha,u,c)<0$, or $\zeta(\alpha,c)>u$. Given Lemma 5, this validates Figure 4 and proves the first statement.

Next, by (9), the expected search time $\tau=N\int_{\ell(\alpha,u,c)}^\infty \pi(\alpha,x)^{N-1}\phi(x)dx$. We show that $\tau_\alpha$ is single-crossing in $u$, namely, negative for lower $u$ and positive for higher $u$.

Now, $\ell_\alpha(\alpha,u,c)$ falls in $u$, and for low enough $u$, $\ell_\alpha(\alpha,u,c)>0$ by (15). Since $\pi_\alpha(\alpha,x,c)<0$ by Claim D.2 (below), we have $\tau_\alpha<0$ for low enough $u$, from (9).

Suppose that $u$ is large so that $\ell_\alpha(\alpha,u,c)<0$. Since $\ell_\alpha(\alpha,u,c)=-[\ell(\alpha,u,c)+\zeta_\alpha(\alpha,c)]/\alpha$ and $\zeta_\alpha<0$ (Lemma 5), $\ell(\alpha,u,c)>0$. Differentiate (9), and then change variables $s=x-\ell(\alpha,u,c)$. Writing $\ell(\alpha,u,c)=\ell$, the slope $\partial\tau/\partial\alpha$ equals:

$$\begin{aligned}&-\ell_\alpha N\pi(\alpha,\ell)^{N-1}\phi(\ell)+N(N-1)\int_0^\infty \pi(\alpha,\ell+s)^{N-2}\pi_\alpha(\alpha,\ell+s)\phi(\ell+s)ds\\ &=N\pi(\alpha,\ell)^{N-1}\phi(\ell)\left(-\ell_\alpha+(N-1)\int_0^\infty\left[\frac{\pi(\alpha,\ell+s)}{\pi(\alpha,\ell)}\right]^{N-1}\frac{\pi_\alpha(\alpha,\ell+s)}{\pi(\alpha,\ell+s)}\frac{\phi(\ell+s)}{\phi(\ell)}ds\right)\end{aligned}$$

Write this as $N\pi(\alpha,\ell)^{N-1}\phi(\ell)\Upsilon(\alpha,u)$. Now, the integrand is negative and rises in $\ell$, because: *(i)* $0>\pi_\alpha(\alpha,\ell+s)/\pi(\alpha,\ell+s)$ rises in $\ell$ by log-supermodularity of $\pi(\alpha,x)$ (Claim D.2); *(ii)* $\pi(\alpha,\ell+s)/\pi(\alpha,\ell)$ falls in $\ell$ by log-concavity of $\pi(\alpha,x)$ in $x$ (Claim D.3); and *(iii)* the $\phi$ ratio falls in $\ell$, as $\phi$ is strictly log-concave. Since $\ell_u(\alpha,u,c)>0$, the integrand rises in $u$. Also, $\ell_\alpha(\alpha,u,c)$ falls in $u$ by (15). As $\Upsilon(\alpha,u)$ is increasing in $u$, $\tau_\alpha(\alpha,u)$ is strictly single-crossing in $u$. But as noted above, $\tau_\alpha(\alpha,u)<0$ for small enough $u$. When $u\to\infty$, $\ell_\alpha(\alpha,u,c)\to-\infty$ and so $\Upsilon(\alpha,u)>0$, and $\tau_\alpha(\alpha,u)>0$. So $\tau_\alpha(\alpha,u)$ changes sign exactly once as $u$ rises from $-\infty$ to $\infty$. $\square$

**Claim D.2** *The attraction $\pi$ obeys $\pi_\alpha(\alpha,x)<0<(\log[\pi(\alpha,x)])_{x\alpha}$.*

Proof: Since $\partial[\zeta(\alpha,c)/\sqrt{1-\alpha^2}]/\partial\alpha < 0$ by Lemma 5, differentiating (33) yields $\pi_\alpha < 0$. Next, rewrite (33) as $\pi(\alpha,x) = \int_{-\infty}^{\infty}\phi(x+s)f(\alpha,s)ds$, where

$$f(\alpha,s) \equiv \Phi\left((\zeta(\alpha,c)-\alpha s)/\sqrt{1-\alpha^2}\right) \text{ for } s>0 \quad \text{ and } \quad f(\alpha,s)\equiv 1 \text{ for } s\leq 0.$$

Since the Gaussian density satisfies $\phi'(x) = -x\phi(x)$, we have

$$-\frac{\partial\log[\pi(\alpha,x)]}{\partial x} = \frac{\int_{-\infty}^{\infty}(x+s)\phi(x+s)f(\alpha,s)ds}{\int_{-\infty}^{\infty}\phi(x+s)f(\alpha,s)ds} = \frac{\int_{-\infty}^{\infty}r\phi(r)f(\alpha,r-x)dr}{\int_{-\infty}^{\infty}\phi(r)f(\alpha,r-x)dr}. \tag{35}$$

This is the mean of a r.v. $R$ with density $\phi(r)f(\alpha,r-x)$. Next, we argue $f(\alpha,s)$ is log-submodular, or equivalently $f(\alpha,s_2)/f(\alpha,s_1)$ falls in $\alpha$ for all $s_2 > s_1$. First, if $s_1,s_2<0$, we have $f(\alpha,s_2)/f(\alpha,s_1)\equiv 1/1 = 1$ weakly falls in $\alpha$. Second, if $s_1,s_2>0$, then $f(\alpha,s_i)\equiv\Phi((\zeta(\alpha,c)-\alpha s_i)/\sqrt{1-\alpha^2})$ for $i=1,2$. Here, it thus suffices that

$$\frac{\partial\log[f(\alpha,s)]}{\partial s} = -\frac{\alpha s}{\sqrt{1-\alpha^2}}\phi\left(\frac{\zeta(\alpha,c)-\alpha s}{\sqrt{1-\alpha^2}}\right)\Big/\Phi\left(\frac{\zeta(\alpha,c)-\alpha s}{\sqrt{1-\alpha^2}}\right). \tag{36}$$

falls in $\alpha$. This follows since $\Phi$ is log-concave, and because $[\zeta(\alpha,c)-\alpha s]/\sqrt{1-\alpha^2}$ falls in $\alpha$, as $\partial[\zeta(\alpha,c)/\sqrt{1-\alpha^2}]/\partial\alpha<0$ by Lemma 5 and $s>0$. Third, for $s_1\leq 0<s_2$, $f(\alpha,s_2)/f(\alpha,s_1) = f(\alpha,s_2) = \Phi\left(\frac{\zeta(\alpha,c)-\alpha s_2}{\sqrt{1-\alpha^2}}\right)$ falls in $\alpha$. Altogether $f(\alpha,s)$ is log-submodular, and thus the middle term in (35) falls in $\alpha$, or $\partial^2\log[\pi(\alpha,x)]/\partial\alpha\partial x > 0$.

**Claim D.3** *For $x\geq 0$, the attraction $\pi$ obeys $\pi_x(\alpha,x) > 0 > (\log[\pi(\alpha,x)])_{xx}$.*

*Proof:* For the log-concavity of $\pi(\alpha,x)$ in $x$, integrate (33) by parts to get

$$\pi(\alpha,x) = \frac{\alpha}{\sqrt{1-\alpha^2}}\int_0^\infty\phi\left(\frac{\zeta(\alpha,c)-\alpha s}{\sqrt{1-\alpha^2}}\right)\Phi(x+s)ds + \Phi(x)\left(1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right).$$

Then

$$\pi_x(\alpha,x) = \frac{\alpha}{\sqrt{1-\alpha^2}}\int_0^\infty\phi\left(\frac{\zeta(\alpha,c)-\alpha s}{\sqrt{1-\alpha^2}}\right)\phi(x+s)ds + \phi(x)\left(1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right) > 0$$

Since the Gaussian density $\phi$ is hump-shaped and peaks at 0, the RHS falls in $x\geq 0$. In other words, $(\log[\pi(\alpha,x)])_{xx} < 0$ for $x\geq 0$. □

## D.3 Accuracy and Quitting Chance: Proof of Equation (34)

Put $u = \zeta(\alpha, c)$ and $x = \ell(\alpha, u, c)$ in (33). Then $(\partial/\partial\alpha)\pi(\alpha, \ell(\alpha, u, c))$ equals

$$\begin{aligned}
&\int_{\ell(\alpha,u,c)}^{\infty} \phi\left(\frac{u-\alpha s}{\sqrt{1-\alpha^2}}\right)\frac{(\alpha u - s)}{\sqrt{1-\alpha^2}}d\Phi(s) + \frac{\partial \ell(\alpha,u,c)}{\partial\alpha}\phi(\ell(\alpha,u,c))\left[1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right] \\
=\;& \phi(u)\int_{\ell(\alpha,u,c)}^{\infty} \phi\left(\frac{s-\alpha u}{\sqrt{1-\alpha^2}}\right)\frac{(\alpha u - s)}{\sqrt{1-\alpha^2}}ds + \frac{\partial \ell(\alpha,u,c)}{\partial\alpha}\phi(\ell(\alpha,u,c))\left[1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right] \\
=\;& -\frac{\phi(u)}{\sqrt{1-\alpha^2}}\phi\left(\frac{\ell(\alpha,u,c)-u\alpha}{\sqrt{1-\alpha^2}}\right) + \frac{\partial \ell(\alpha,u,c)}{\partial\alpha}\phi(\ell(\alpha,u,c))\left[1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right] \\
=\;& \left(\frac{-u+\zeta(\alpha,c)}{\alpha^2}\right)\phi(\ell(\alpha,u,c))\left[1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right]
\end{aligned}$$

The second equality uses the Gaussian density property $\partial\phi(s)/\partial s = -s\phi(s)$. Since the quitting chance is $q = \pi(\alpha, \ell(\alpha, u, c))^N$ by (12),

$$\frac{\partial q}{\partial\alpha} = N\pi(\alpha,\ell(\alpha,u,c))^{N-1}\left(\frac{-u+\zeta(\alpha,c)}{\alpha^2}\right)\phi\left(\frac{u-\zeta(\alpha,c)}{\alpha}\right)\left[1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right] \quad (37)$$

## D.4 Accuracy Helps the User

**Claim D.4** *The user's marginal benefit of accuracy is positive and single-peaked in $u$.*

Proof: Since $\partial\mathcal{V}/\partial u = q$ by (16), $\partial^2\mathcal{V}/\partial u\partial\alpha = \partial q/\partial\alpha$. We can derive $\partial\mathcal{V}/\partial\alpha$ by integrating (37) over $u$. We use the boundary condition $\partial\mathcal{V}/\partial\alpha|_{u=\infty} = 0$, because as $u \to \infty$ a user never clicks through and the accuracy becomes irrelevant for payoff. Integrating (37) in $u$ yields

$$\frac{\partial\mathcal{V}(\alpha,c,u)}{\partial\alpha} = \left[1-\Phi\left(\frac{\zeta(\alpha,c)}{\sqrt{1-\alpha^2}}\right)\right]\int_{\ell(\alpha,u,c)}^{\infty} N\pi(\alpha,x)^{N-1}x d\Phi(x). \quad (38)$$

Since $\ell(\alpha, u, c) > -\infty$ for any $u > -\infty$, we have $\int_{\ell(\alpha,u,c)}^{\infty} x d\Phi(x) > 0$. As $\pi(\alpha, x)^{N-1}$ is positive and strictly increasing, and $\int_{\ell(\alpha,u,c)}^{\infty} x d\Phi(x) > 0$, the RHS of (38) is positive.[22] Hence, $\partial\mathcal{V}/\partial\alpha > 0$. But the lower support of the integral in (38) rises in $u$. As the integrand is first negative and then positive, $\partial\mathcal{V}/\partial\alpha$ rises and then falls in $u$. □

[22] By Karlin and Rubin (1955), if $f$ is single-crossing and $\int f(x)dx > 0$, positivity is preserved if one multiplies the integrand by a positive and increasing function $b(x)$, namely $\int f(x)b(x)dx > 0$.

## D.5 Value of Search Engine: Proof of Proposition 2

A. Changing Outside Option. Let $q_\alpha$ be the value of $q$ when the accuracy is $\alpha$. By (16),

$$\frac{\partial[\mathcal{V}(\alpha,c,u)-\mathcal{V}(0,c,u)]}{\partial u}=q_\alpha-q_0.$$

To show that the value of search engine rises/falls in $u$ as $\zeta(0,c)\gtrless u$, we prove a reverse single crossing property: $q_\alpha - q_0 \lesseqgtr 0$ when $\zeta(0,c)\lesseqgtr u$.

Let $\zeta(0,c)<u$. If $\alpha=0$ (so no known factor), the user never clicks, and elects his outside option with chance $q_0=1$. So $q_{\alpha'}-q_0\leq 0$ for all $\alpha'>0$. Posit $\zeta(0,c)\geq u$. If $\alpha=0$, the user searches every period, choosing his outside option if all $N$ searches fail. So $q_0=\Phi^N(u)$, by prospective independence. For $\alpha>0$, the user exercises the outside option at least if it is best (chance $\Phi^N(u)$). So $q_\alpha\geq\Phi^N(u)$, and $q_\alpha-q_0\geq 0$.

Next, $\Pi(\alpha,c,u)=\mathcal{V}(\alpha,c,u)-\mathcal{V}(0,c,u)\to 0$ since $\mathcal{V}(\alpha,c,u)-u\to 0$ as $u\to\infty$ for all $\alpha$. When $u=-\infty$, we have $\mathcal{V}(\alpha,c,-\infty)=E[\alpha X_b+\sqrt{1-\alpha^2}Z_b]-\tau c-\kappa$ and $\partial\mathcal{V}/\partial\alpha|_{\alpha=0}=E[X_b]$. Also, $E[X_b]>0$ when $\alpha=0$, as a user is likely to exercise an earlier web site and the known factors are sorted in descending order and have zero expected value. So $\Pi_\alpha(\alpha,c,-\infty)>0$ at $\alpha=0$, and $\Pi(\alpha,c,-\infty)>0$ for all $\alpha>0$.

B. Changing Clicking Cost. Equation (16) yields:

$$\frac{\partial[\mathcal{V}(\alpha,c,u)-\mathcal{V}(0,c,u)]}{\partial c}=\tau_0-\tau_\alpha.$$

Now we argue that the value of search engine is single peaked in $c$. First, as $c\to 0$, $\zeta(\alpha,c)\to\infty$, by (14), and thus the cutoff $\bar{w}_{n+1}\to\infty$ for all $n$, by Lemma 3. In the limit, the user clicks every web site, and so $V(\alpha,c,u)\to E[\max\{u,W_1,W_2,\ldots,W_N\}]$.

Next, as $c\to\infty$, $\zeta(\alpha,c)\to-\infty$, and so the cutoff $\bar{w}_{n+1}\to-\infty$ for all $n$, by Lemma 3. In the limit, the user never clicks a web site, and thus $V(\alpha,c,u)\to u$.

Since $\zeta(0,c)$ is continuous in $c$, and $\zeta(0,c)\to\pm\infty$ respectively as $c\downarrow 0$ and $c\uparrow\infty$, and as $\partial\zeta(0,c)/\partial c<0$, there exists $\hat{c}>0$ such that $\zeta(0,c)<u$ iff $c>\hat{c}$. If $c>\hat{c}$, the user never clicks, and his search time is $\tau_0=0$; hence, $\tau_0-\tau_\alpha\leq 0$. Suppose $c<\hat{c}$. When $\alpha=0$, the user stops at stage $n<N$ iff the current web site payoff exceeds $\bar{w}_{n+1}=\zeta(0,c)$, namely, if $z_n>\zeta(0,c)$, recalling Lemma 3. When $\alpha>0$, the user stops if $\alpha x_n+\sqrt{1-\alpha^2}z_n\geq\alpha x_{n+1}+\zeta(\alpha,c)$. Since $x_n>x_{n+1}$, a user accepts web site $n$ if $z_n\geq\zeta(\alpha,c)/\sqrt{1-\alpha^2}$. Since $\zeta(\alpha,c)/\sqrt{1-\alpha^2}$ falls in $\alpha$, we have $\zeta(\alpha,c)/\sqrt{1-\alpha^2}<\zeta(0,c)$, and so a user is more likely to stop if $\alpha>0$ than if $\alpha=0$. Hence, $\tau_0-\tau_\alpha\geq 0$. All told, the search engine value is single-peaked in the cost $c$. $\square$

## D.6 Search Engine Synergies: Proof of Proposition 3

First, the number of options $N$ and accuracy $\alpha$ are complements. To see this, consider the slope $\partial\mathcal{V}/\partial\alpha$. This exists by (38), and is positive. Also, differencing (38) in $N$:

$$\frac{\partial\mathcal{V}}{\partial\alpha}(N+1) - \frac{\partial\mathcal{V}}{\partial\alpha}(N) \propto \int_{\ell(\alpha,u,c)}^{\infty} \pi(\alpha,x)^N[(N+1)\pi(\alpha,x) - N]x\phi(x)dx. \tag{39}$$

Now, $[(N+1)\pi(\alpha,x) - N]$ transitions negative to positive exactly once as $x$ increases from $-\infty$ to $\infty$, as $\pi_x(\alpha,x) > 0$ by Claim D.3. Also, $\pi(\alpha,-\infty) = 0$ and $\pi(\alpha,\infty) = 1$ by (33), since $\ell(\alpha,u,c)$ rises from $-\infty$ to $\infty$ as $u$ rises from $-\infty$ to $\infty$, given (15).

Given the $x$ factor, the integrand has 0 or 2 sign changes. In the first case, we are done. Otherwise, the sign sequence is $+,-,+$ as $x$ rises from $-\infty$ to $\infty$. Since $\ell_u(\alpha,u,c) > 0$, the difference (39) first falls in $u$, then rises, and finally falls. If (39) vanishes as $u \to \infty$ (true, as $\ell_u(\alpha,u,c)$ explodes), and is positive at the local minimum $u = \bar{u}$ where the integrand shifts from falling to rising, then (39) is always positive.

Now, $u$ only affects the integral in (39) via the lower support $\ell(\alpha,u,c)$. As its minimum is at $u = \bar{u}$, and $\ell_u(\alpha,u,c) > 0$, the integrand $[(N+1)\pi(\alpha,x)-N]x$ changes from $+$ to $-$ at $x = \ell(\alpha,\bar{u},c)$. Since $[(N+1)\pi(\alpha,x)-N]$ and $x$ single-cross in $x$, their product's first sign change is at $x = \ell(\alpha,\bar{u},c)$. There are only two possible cases:

Case 1: $(N+1)\pi(\alpha,\ell(\alpha,\bar{u},c)) = N$ and $\ell(\alpha,\bar{u},c) < 0$. Since $(N+1)\pi(\alpha,x)-N$ rises in $x$ by Claim D.3, we have $(N+1)\pi(\alpha,x) - N > 0$ for $x \geq \ell(\alpha,\bar{u},c)$. By Claim D.4, $\partial\mathcal{V}/\partial\alpha > 0$. By (38), $\int_{\ell(\alpha,\bar{u})}^{\infty} \pi(\alpha,x)^N x\phi(x)dx > 0$. Since the integrand single-crosses in $x$, the integral remains strictly positive when the integrand is scaled by the increasing function $b(x) = (N+1)\pi(\alpha,x) - N > 0$ on $[\ell(\alpha,\bar{u}),\infty)$, as noted in footnote 22. Altogether, the integral in (39) is strictly positive at $u = \bar{u}$.

Case 2: $(N+1)\pi(\alpha,\ell(\alpha,\bar{u},c)) < N$ and $\ell(\alpha,\bar{u},c) = 0$. Since the expected time $\tau$ increases in $N$ by (the text result after) Theorem 6, its definition in (9) yields:

$$\int_{\ell(\alpha,\bar{u})}^{\infty} \pi(\alpha,x)^{N-1}[(N+1)\pi(\alpha,x) - N]\phi(x)dx = \tau(N+1) - \tau(N) > 0.$$

This integrand is single-crossing in $x$, and is the integrand in (39) times $1/x$. Since $x \geq \ell(\alpha,\bar{u},c) = 0$, the integral in (39) is positive, by Karlin and Rubin (1955). $\square$

## D.7 Proofs for Sequential Web Search in §8.3

**Claim D.5** *Posit* limit $(\star)$*: the outside option $u$ explodes ($u \uparrow \infty$), and the clicking cost vanishes ($c \downarrow 0$) but the CTR holds constant. Then* (*a*) *the limit coefficient $\beta_1$ is positive, and* (*b*) *the coefficient $\beta_2$ tends to* 0.

Proof of (*a*): Let $\mathbb{E}_C$ be the *click-through event* $\mathcal{X}_1 > u - \zeta(c)$ or $T \geq 1$, by (15). The OLS sample estimate of $\beta_1$ is $\hat{\beta}_1 = \mathrm{Cov}_e(T, W_1|\mathbb{E}_C)/\mathrm{Var}_e(W_1|\mathbb{E}_C)$, where $\mathrm{Cov}_e(T, W_1|\mathbb{E}_C)$ and $\mathrm{Var}_e(W_1|\mathbb{E}_C)$ are the sample covariance and variance given $\mathbb{E}_C$. Then $\hat{\beta}_1$ converges in probability to $\beta_1 = \mathrm{Cov}(W_1, T|\mathbb{E}_C)/\mathrm{Var}(W_1|\mathbb{E}_C)$ as $N \uparrow \infty$.

Since the cdf of $\mathcal{X}_1$ is $P(\mathcal{X}_1 \leq x_1) = G(x_1)^N$, the conditional expectation

$$E[W_1|\mathbb{E}_C] = \int_{u-\zeta(c)}^{\infty} \int_{-\infty}^{\infty} (x_1 + z_1) dH(z_1) dG(x_1)^N / [1 - G(u - \zeta(c))^N]$$

is constant as $u \to \infty, c \downarrow 0$, fixing $u - \zeta(c) = \bar{\ell}$ (limit $(\star)$). Similarly, since $\mathrm{Var}(W_1|\mathbb{E}_C) = E[W_1^2|\mathbb{E}_C] - E[W_1|\mathbb{E}_C]^2$, the limit variance only depends on $\bar{\ell}$. So the sign of $\beta_1$ in this limit depends on $\mathrm{Cov}(W_1, T|\mathbb{E}_C) > 0$. Let $t(x_1, z_1, u, c)$ be the expected number of searches when the user clicks through if $\mathcal{X}_1 = x_1$ and $\mathcal{Z}_1 = z_1$. Then $\mathrm{Cov}(W_1, T|\mathbb{E}_C) = \mathrm{Cov}(W_1, t(\mathcal{X}_1, \mathcal{Z}_1, u, c)|\mathbb{E}_C)$. We derive a formula for $t(x_1, z_1, u, c)$.

Assume $\mathcal{X}_1 = x_1$ and $\mathcal{Z}_1 = z_1$. By Lemmas 1 and 3, the user enters stage $n$ iff $\mathcal{X}_n + \zeta(c) > \Omega_n = \max(u, w_1, w_2, \ldots, w_n)$. In the limit $u \to \infty$ and $c \to 0$, and so $\zeta(c) \to \infty$, the condition becomes $\mathcal{X}_n > u - \zeta(c) = \bar{\ell}$. By the Markov property of order statistics (footnote 8), the distribution of the known factors of the remaining $N-1$ web sites is the same as $N-1$ *i.i.d.* draws from cdf $G(x)/G(x_1)$ for $x < x_1$. So in limit limit $(\star)$, a randomly selected option in the subgame is clicked iff its known factor exceeds $\bar{\ell}$, which occurs with chance $[1 - G(\bar{\ell})/G(x_1)]$. Since each of the $N-1$ options is independently clicked with chance $[1 - G(\bar{\ell})/G(x_1)]$, the expected number of searches in the limit $u \to \infty, c \to 0$ is $(N-1)[1 - G(\bar{\ell})/G(x_1)]$. Fixing $\bar{\ell}$,

$$\lim_{c \to 0, u \to \infty} t(x_1, z_1, u, c) = 1 + (N-1)[1 - G(\bar{\ell})/G(x_1)] \equiv \bar{t}(x_1). \tag{40}$$

Altogether, $\mathrm{Cov}(W_1, T|\mathbb{E}_C) \to \mathrm{Cov}(\mathcal{X}_1 + \mathcal{Z}_1, \bar{t}(\mathcal{X}_1)|\mathbb{E}_C)$ at the limit. Since $\mathcal{X}_1$ is independent of $\mathcal{Z}_1$ *even* given $\mathbb{E}_C$, $\mathrm{Cov}(\mathcal{X}_1 + \mathcal{Z}_1, \bar{t}(\mathcal{X}_1)|\mathbb{E}_C) = \mathrm{Cov}(\mathcal{X}_1, \bar{t}(\mathcal{X}_1)|\mathbb{E}_C)$. Finally, $\mathrm{Cov}(\mathcal{X}_1, \bar{t}(\mathcal{X}_1)|\mathbb{E}_C) > 0$ as $\bar{t}(x_1)$ strictly rises in $x_1$ by (40). Altogether, the coefficient $\beta_1 = \mathrm{Cov}(W_1, T|\mathbb{E}_C)/\mathrm{Var}(W_1|\mathbb{E}_C) > 0$ as $u \to \infty$ and $c \to 0$. □

Proof of (*b*): As $N$ explodes, the OLS estimate $\hat{\beta}_2$ converges in probability to $\mathrm{Cov}(T, \mathcal{Z}_1|\mathbb{E}_C)/\mathrm{Var}(\mathcal{Z}_1|\mathbb{E}_C)$. As $\mathcal{X}$ and $\mathcal{Z}$ factors are independent, the conditional expectation of $\mathcal{Z}_1$ has cdf $H$ under $\mathbb{E}_C$. All told, $\text{Var}(\mathcal{Z}_1|\mathbb{E}_C) = \text{Var}(\mathcal{Z}) > 0$ as $N \to \infty$.

If $(\mathcal{X}_1, \mathcal{Z}_1) = (x_1, z_1)$, then the limit $\bar{t}(\mathcal{X}_1)$ of expected search times $t(x_1, z_1, u, c)$ as $u \to \infty$ and $c \to 0$ is constant in $z_1$, by (40). As $t(x_1, z_1, u, c) \leq N - 1$, the Dominated Convergence Theorem implies that $\text{Cov}(t(\mathcal{X}_1, \mathcal{Z}_1, u, c), \mathcal{Z}_1|\mathbb{E}_C) \to \text{Cov}(\bar{t}(\mathcal{X}_1), \mathcal{Z}_1|\mathbb{E}_C)$. So $\beta_2 \to \text{Cov}(\bar{t}(\mathcal{X}_1), \mathcal{Z}_1|\mathbb{E}_C)/\text{Var}(\mathcal{Z}) = 0$, as $\mathcal{X}_1$ and $\mathcal{Z}_1$ are independent on $\mathbb{E}_C$. $\square$

**Claim D.6** *If $h(z)$ has a thin tail, then $\beta_3$ has a non-negative limit given* $(\star)$.

*Proof:* Let $\mathbb{E}_P$ be the event that the user eventually purchases. By OLS, $\beta_3 = \text{Cov}(T, \mathcal{X}_1|\mathbb{E}_P)/\text{Var}(\mathcal{X}_1|\mathbb{E}_P)$, where $\text{Cov}(T, \mathcal{X}_1|\mathbb{E}_P)$ and $\text{Var}(\mathcal{X}_1|\mathbb{E}_P)$ are the covariance and variance. Then $\beta_3$ is non-negative provided $\text{Cov}(T, \mathcal{X}_1|\mathbb{E}_P) \geq 0$ in the limit $(\star)$.

The user clicks through if he buys, and buys if he clicks through *and* $\mathcal{X}_1 + \mathcal{Z}_1 > u$. So $P(\{\mathcal{X}_1+\mathcal{Z}_1 > u\} \cap \{\mathbb{E}_P\}) = P(\{\mathcal{X}_1+\mathcal{Z}_1 > u\} \cap \{\mathbb{E}_C\}) = \int_{\bar{\ell}}^{\infty} [1 - H(u - x_1)] dG(x_1)^N$. Since $P(\mathbb{E}_P) = 1 - q = 1 - \pi(u - \zeta(c), c)^N$ by (11), Bayes rule gives:

$$P(\mathcal{X}_1+\mathcal{Z}_1 > u|\mathbb{E}_P) = \frac{\int_{\bar{\ell}}^{\infty} [1 - H(u - x_1)] dG(x_1)^N}{1 - \pi(\bar{\ell}, c)^N} = \int_0^{\infty} \left[\frac{1 - H(\zeta(c) - s)}{1 - \pi(\bar{\ell}, c)^N}\right] dG(s+\bar{\ell})^N.$$

By (6)–(7), the limit as $\zeta(c) \to \infty$ as $c \to 0$ of the bracketed term in the integrand is

$$\lim_{\zeta(c)\to\infty} \frac{1 - H(\zeta(c) - s)}{1 - [\int_0^{\infty} g(\bar{\ell} + r) H(\zeta(c) - r)\, ds + G(\bar{\ell})]^N} = \lim_{\zeta(c)\to\infty} \frac{h(\zeta(c) - s)}{N \int_0^{\infty} g(\bar{\ell} + r) h(\zeta(c) - r)\, dr}$$

by l'Hopital's rule, since $\lim_{\zeta(c)\to\infty} [\int_0^{\infty} g(\bar{\ell}+r) H(\zeta(c) - r)\, ds + G(\bar{\ell})] = 1$. In limit $(\star)$:

$$\lim_{c\to 0} P(\mathcal{X}_1 + \mathcal{Z}_1 > u|\mathbb{E}_P) = \lim_{\zeta(c)\to\infty} \frac{\int_0^{\infty} G(s + \bar{\ell})^{N-1} g(s + \bar{\ell}) h(\zeta(c) - s)\, ds}{\int_0^{\infty} g(r + \bar{\ell}) h(\zeta(c) - r)\, dr}. \tag{41}$$

Write (41) as $\lim_{\zeta(c)\to\infty} E[G(S+\bar{\ell})^{N-1}]$, where the r.v. $S$ has density $g(s+\bar{\ell}) h(\zeta(c) - s)$. Since $h$ has a thin tail, as $\zeta(c) \to \infty$ in the limit $(\star)$, $h(\zeta(c) - s_1)/h(\zeta(c) - s_2) \to 0$ for $s_1 < s_2$ by Claim C.2, whence $E[G(S+\bar{\ell})^{N-1}] \to 1$, and so $P(\mathcal{X}_1+\mathcal{Z}_1 > u|\mathbb{E}_P) \to 1$.

In the limit $(\star)$, since $\lim P(\mathcal{X}_1 + \mathcal{Z}_1 > u|\mathbb{E}_P) = 1$, we have $\text{Cov}(T, \mathcal{X}_1|\mathbb{E}_P) - \text{Cov}(T, \mathcal{X}_1|\{\mathcal{X}_1+\mathcal{Z}_1 > u\} \cap \mathbb{E}_P) \to 0$. Next, $\{\mathcal{X}_1+\mathcal{Z}_1 > u\} \cap \mathbb{E}_P = \{\mathcal{X}_1+\mathcal{Z}_1 > u\} \cap \mathbb{E}_C$, as $\mathbb{E}_P \subset \mathbb{E}_C$, while $\{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_C$ implies $\{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_P$, as the user eventually purchases if he clicks through and the first website dominates the outside option. So $\text{Cov}(T, \mathcal{X}_1|\{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_P) = \text{Cov}(T, \mathcal{X}_1|\{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_C)$, i.e.

$$\lim \text{Cov}(T, \mathcal{X}_1|\mathbb{E}_P) = \lim \text{Cov}(T, \mathcal{X}_1|\{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_C) \quad \text{in the limit } (\star) \tag{42}$$

Given $\mathbb{E}_C$, the expected unconditional search time $T$ is the expectation of $t(\mathcal{X}_1, \mathcal{Z}_1, u, c)$, i.e. the mean number of searches when $\mathcal{X}_1 = x_1$, $\mathcal{Z}_1 = z_1$ and the user clicks through:

$$\operatorname{Cov}(T, \mathcal{X}_1 | \{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_C) = \operatorname{Cov}(t(\mathcal{X}_1, \mathcal{Z}_1, u, c), \mathcal{X}_1 | \{\mathcal{X}_1 + \mathcal{Z}_1 > u\} \cap \mathbb{E}_C). \quad (43)$$

By equation (40), $t(x_1, z_1, u, c) \to \bar{t}(x_1)$ in limit $(\star)$, which also rises in $x$. Then $\lim \operatorname{Cov}(t(\mathcal{X}_1, \mathcal{Z}_1, u, c), \mathcal{X}_1 | \{\mathcal{X}_1 + \mathcal{Z}_1 > u\}, \mathbb{E}_C) = \lim \operatorname{Cov}(\bar{t}(\mathcal{X}_1), \mathcal{X}_1 | \{\mathcal{X}_1 + \mathcal{Z}_1 > u\}, \mathbb{E}_C) \geq 0$. So by (42)–(43), $\lim \beta_3 = \lim \operatorname{Cov}(T, \mathcal{X}_1 | \mathbb{E}_P) / \operatorname{Var}(\mathcal{X}_1 | \mathbb{E}_P) \geq 0$ in the limit $(\star)$. $\square$

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*Online Appendix for*
*"Optimal Sequential Search Among Alternatives"*

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# A The Value Function

**Claim A.1** *As the search cost $c$ rises, the value function $V_n(\Omega_n)$ grows weakly steeper.*

PROOF: After the last draw, the terminal value function is $V_N(\Omega) = \Omega$, and so the claim is trivially true. Suppose the claim holds at stage $n+1$. If $\Omega_n > \bar{w}_{n+1}$, then the searcher stops at stage $n$ and $V_n'(\Omega_n) = 1$. Thus $V_n'(\Omega_n)$ remains unchanged as $c$ rises. If $\Omega_n < \bar{w}_{n+1}$, then $V_n'(\Omega_n+) = F_{n+1}(\Omega_n)V_{n+1}'(\Omega_n+)$ by (4). Since $V_{n+1}'(\Omega_n+)$ rises in $c$, so does $V_n'(\Omega_n+)$. If $\Omega_n = \bar{w}_{n+1}$, then $V_n'(\Omega_n+)$ rises from $F_{n+1}(\Omega_n)V_{n+1}'(\Omega_n+)$ to 1 because $\bar{w}_{n+1}$ falls in $c$ and $V_{n+1}'(\Omega_n+) \leq 1$ by Lemma 2. In all cases $V_n'(\Omega_n+)$ rises in $c$ and thus claim holds at $n$. Inductively, the claim is always true. $\square$

By Lemma 2, the slope $V_n'(\Omega)$ in (4) is the chance that the best-so-far $\Omega$ will be eventually exercised. In the same spirit, now we show that the derivative of $-V_n(\Omega)$ with respect to the search cost $c$ equals the expected number of remaining searches.

**Claim A.2 (Pre-Query Value Differentiability)** *The value function $V_n(\Omega_n)$ at period $n$ is differentiable in the search cost $c$ when the stage $n$ best option so far $\Omega_n \neq \bar{w}_{j+1}$ for $j \in \{n+1, \ldots, N\}$. The derivative $-\partial V_n(\Omega_n)/\partial c$ is the expected number of remaining searches.*

PROOF: Assume $\Omega_n \neq \bar{w}_{j+1} = \mathcal{X}_{j+1} + \zeta(c)$ for $j \in \{i, \ldots, N\}$. After the last draw, the terminal value function is $V_N(\Omega_N) = \Omega_N$. Since $\partial V_N(\Omega_N)/\partial c = 0$, all claims are true at stage $N$. Suppose the statements hold for stage $n+1$. At stage $n$, if $\Omega_n > \bar{w}_{n+1}$ then the searcher stops searching. By (4), we have $V_n(\Omega_n) = \Omega_n$ on $[\bar{w}_{n+1}, \infty)$ and so $\partial V_n(\Omega_n)/\partial c = 0$. If $\Omega_n < \bar{w}_{n+1}$, then the searcher continues to stage $n+1$. In this case, $-\partial V_n(\Omega_n)/\partial c = 1 - [\partial V_{n+1}(\Omega_n)/\partial c]F_{n+1}(\Omega_n) - \int_{\Omega_n}^{\infty} [\partial V_{n+1}(z)/\partial c] dF_{n+1}(z)$ by (4). The integral exists because $\partial V_{n+1}(z)/\partial c$ exists except at finite number of points. The value of $-\partial V_n(\Omega_n)/\partial c$ equals 1 plus the expected number of remaining searches. Inductively, the statement holds for all stages. $\square$

Next, we show that the ex ante value $\mathcal{V}$ is differentiable in $u$ and $c$ as long as $\mathcal{X}$ is non-degenerate. Consider the stage after the realization of $\vec{\mathcal{X}} \equiv \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ but before the searcher explores any option. Let $V_0(u, c, \vec{x})$ be the searcher's expected payoff. By Lemma 2 and Claim A.2, $V_0(u, c, \vec{x})$ is differentiable in $u$ and $c$ except when $u = x_j + \zeta(c)$ for any $j \in \{1, \ldots, N\}$. The searcher's ex ante payoff before the realization of the known factors is $\mathcal{V}(u, c) = E[V_0(u, c, \vec{\mathcal{X}})]$ where the expectation is taken over $\vec{\mathcal{X}}$. Both $\partial\mathcal{V}(u, c)/\partial u$ and $\partial\mathcal{V}(u, c)/\partial c$ exist as the cases where $\partial V_0(u, c, \vec{\mathcal{X}})/\partial u$ and $\partial V_0(u, c, \vec{\mathcal{X}})/\partial c$ do not exist are of measure zero.

The slope $\partial V_0(u, c, \vec{x})/\partial u$ is the quitting chance given $\vec{x}$ by Lemma 2. Similarly, $-\partial V_0(u, c, \vec{x})/\partial c$ is the expected search time given $\vec{x}$ by Claim A.2. Thus (16) is true, namely $\partial\mathcal{V}(u, c)/\partial u = q$ and $\partial\mathcal{V}(u, c)/\partial c = -\tau$. □

# B Hazard Rates

The initial quitting chance $q_0 = G(u - \zeta(c))^N$ rises in $u$ and $c$ because $\zeta'(c) < 0$ by (15). Later ex ante quitting chances $q_n \approx 0$ for any $1 \leq i < N$, for all small enough $c \geq 0$ and large enough $c < \infty$, because the searcher never stops at an intermediate stage $n$ when search is very cheap or prohibitively costly.

**Claim B.1** *The stage $n$ quitting chance $q_n$ is single-peaked in $c$ and $u$, for $1 \leq n < N$, and $q_N$ falls in $c$.*

PROOF: Consider (10). First, $G(u - \zeta(c))$ is log-concave in $\zeta(c)$, since $G$ is log-concave. Also, since we have assumed that $H$ and $g$ are log-concave, $\delta(u - \zeta(c), c) = \int_0^\infty H\left(\zeta(c) - r\right) g(r + u - \zeta(c))dr$ is log-concave in $\zeta(c)$ by Prekopa's Theorem.[23] since log-concavity is preserved under multiplication, $q_n$ is log-concave in $\zeta(c)$ and thus is single-peaked in $\zeta(c)$ for all $1 \leq n < N$. Because $\zeta'(c) < 0$, $q_n$ is single-peaked in $c$. Similarly, $G(u - \zeta(c))$ and $\delta(u - \zeta(c), c)$ are also log-concave in $u$, thus $q_n$ is single-peaked in $u$ for all $1 \leq i < N$. Finally, (6) implies that $\delta(u - \zeta(c), c) = \int_{u-\zeta(c)}^\infty H\left(u - r\right) dG(r)$ falls in $c$ and $u$, and thus so too does $q_N = \delta(u - \zeta(c), c)^N$. □

**Claim B.2** *The interim striking hazard rate $\bar{\mathcal{K}}_n(x_n, c)$ rises in $c$ and $x_n$.*

PROOF: Since $\zeta'(c) < 0$, the integrand of (28) rises in $c$, since the bracketed terms do. As the integrand in (28) is $h(\zeta(c))$ if $z_n = \zeta(c)$, its partial derivative in $c$ from the Fundamental Theorem of Calculus is $\zeta'(c)h(\zeta(c))$. This cancels with $-\partial H(\zeta(c))/\partial c$ in differentiating $\bar{\mathcal{K}}_n(x_n, c) = 1 - H(\zeta(c)) + A(x_n, c)$. So $\bar{\mathcal{K}}_n(x_n, c)$ rises in $c$.

[23] By Prekopa's Theorem, log-concavity is preserved under partial integration. See An (1997).

Next, $\bar{\mathcal{K}}_n(\chi_n, c)$ rises in $\chi_n$. It suffices that $A(\chi_n, c)$ rises in $\chi_n$. To this end,

$$A(\chi_n, c)=\int_{u-\chi_n}^{\zeta(c)} h(z_n) \left[\frac{\int_0^\infty \frac{H(z_n-t)}{H(\zeta(c)-t)} H(\zeta(c)-t)g(t+\chi_n)dt}{\int_0^\infty H(\zeta(c)-t)g(t+\chi_n)dt}\right]^{n-1} \left[\frac{G(\chi_n+z_n-\zeta(c))}{G(\chi_n)}\right]^{N-n} dz_n.$$

changing variables $r=t+\chi_n$ in (28). The first bracketed term is $E[H(z_n-T)/H(\zeta(c)-T)]$, for the new r.v. $T$ with density $H(\zeta(c)-t)g(t+\chi_n)$. As $g$ is log-concave, $T$ falls stochastically in $\chi_n$. The first bracketed term rises in $\chi_n$ as $H(z_n-T)/H(\zeta(c)-T)$ falls in $T$, as $H$ is log-concave and $z_n < \zeta(c)$. The second bracketed term rises, given $G$ log-concave and $z_n < \zeta(c)$. As the integral support rises in $\chi_n$, so does $A(\chi_n, c)$. $\square$

**Claim B.3 (Striking Hazard Rate)** *$\mathcal{K}_n$ rises in $c$ and falls in $u$ .*

Proof: First, $\mathcal{K}_n = E_{\mathcal{X}_n}[\bar{\mathcal{K}}_n(\mathcal{X}_n, c)]$ rises in $c$ as $\bar{\mathcal{K}}_n(\chi_n, c)$ rises in $\chi_n$ and $c$ by Lemma B.2 and $\mathcal{X}_n$ rises stochastically in $c$ by Lemma 4: For the direct effect of greater search costs $c$ and selection bias on $\chi_n$ reinforce.

Second, as $g(s+\chi)/g(\chi)$ falls in $\chi$ by log-concavity, and $\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt$ is log-submodular in $(s, \chi)$ by log-concavity of $H$ and $g$, Claim B.6's bracketed integral falls in $\mathcal{X}_n$. As $\mathcal{X}_n$ stochastically rises in $u$ (Lemma 4), $\mathcal{K}_n$ falls in $u$. $\square$

**Claim B.4 (Recall and Quitting Hazard Rate)** *$\mathcal{R}_n^N$ falls and $\mathcal{Q}_n^N$ rises in $u$.*

Proof: As $B(\chi, n)$ falls in $\chi$ by Claim B.7, $\mathcal{R}_n^N = E_{\mathcal{X}_n}[B(\mathcal{X}_n, n)]$ falls in $u$ because $\mathcal{X}_n$ rises stochastically in $u$ by Lemma 4.

Recall that $\mathcal{S}_n = \mathcal{Q}_n + \mathcal{K}_n + \mathcal{R}_n$. Since $\mathcal{X}_n$ stochastically rises in $u$ by Lemma 4, the stopping hazard rate $\mathcal{S}_n$ rises in $u$ by (22). Since $\mathcal{S}_n$ rises in $u$ and $\mathcal{K}_n$ and $\mathcal{R}_n$ fall in $u$, by Claims B.3 and B.4, $\mathcal{Q}_n$ rises in $u$. $\square$