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Econ 713 Midterm

(Tuesday, February 15, 2021 in Class)

Cite any theorems you apply. Rigorously justify everything.

There are 75 total points in this exam, but 10 of the points are bonus. Enjoy!

Part I: NTU Matching ($20 = 10 + 10$ Points)

1) Four dentists want to match with four interns, their payoffs of matching are given in the grid below. The payoff of the outside option is zero.

	I1	I2	I3	I4
D1	3,3	2,4	4,1	1,2
D2	4,2	3,3	2,4	1,3
D3	4,1	3,2	2,3	1,4
D4	2,4	4,1	3,2	1,1

a) Assuming dentists propose to the interns, what are the resulting matches from the Gale-Shapley algorithm? How many rounds does it take to complete?

Solution: In the first round, D1 proposes to I3, D2 proposes to I1, D3 proposes to I1, and D4 proposes to I2. I1 rejects D3, who proposes to I2. I2 rejects D4, who proposes to I3. I3 rejects D1, who proposes to I1. I1 rejects D2, who proposes to I2. I2 rejects D3, who proposes to I3. I3 rejects D4, who proposes to I1. I1 rejects D1, who proposes to I2. I2 rejects D2, who proposes to I3. I3 rejects D3, who proposes to D4 and the algorithm ends. In total this takes $10 = 4^2 - 2(4) + 2$ *rounds (the maximum number of rounds for a* 4×4*). Our matches are (D1,I2), (D2,I3), (D3,I4), and (D4,I1).*

b) Assume that interns propose to the dentists. Who matches with whom after the Gale-Shapley algorithm. How many rounds does the algorithm it take?

Solution: I1 proposes to D4, I2 proposes to D1, I3 proposes to D2, and I4 proposes to D3. The game ends after a single round.

c) Find all possible stable matchings.

Solution: The matches are the same with interns proposing (dentist pessimal stable match) and dentists proposing (dentist optimal stable match) so this is the unique stable match. None others exist.

2) [Slightly Tricky] Assume the interns cannot afford to live alone and thus must pair off into 2 pairs of roommates. Assume the nontransferable money payoffs of matches are given by the grid below.

	I1	I2	I3	I4
I1		4,3	3,4	2,4
I2	3,4		4,3	2,3
I3	4,3	3,4		2,2
I4	4,2	3,2	2,2	

Either find a stable set of roommate pairing, or show that none exists.

Hint: Who matches with I4?

Answer: In this case, notably, there does not exist a stable match. Suppose, for contradiction, that one exists. Then, one of I1,I2, or I3 would have to pair with I4. Then, whichever intern is paired with I4 will strictly prefer to leave and form a new matching with any other intern. By inspection, no matter whether the non-I4 matching is (I1,I2), (I1,I3), or (I2,I3), there will always be a strictly profitable deviation for one of the members of this pair to leave and form a new match with the intern formally paired with I4. Thus, no stable match can exist.

Part II: TU Matching ($35 = 15 + 10 + 10$ points)

3) A unit mass of assistants and managers have indices a and m distributed uniformly on the interval $[0, 1]$. The outside option to matching is so low such that everyone will match. Assume transferable utility and match payoffs $f(a, m) = \log(1+a+m)$.

   a) Find the efficient allocation of assistants to managers.

      Solution: The production function f is submodular $\left(\frac{\partial^2 f}{\partial a \partial m} = -\frac{1}{(1+a+m)^2} < 0\right)$. So NAM is efficient. Thus, manager m pairs with assistant $a(m) = 1 - m$.

   b) What is the slope of the competitive equilibrium wage of a manager in her type, and of the wage of an assistant in his type?

      Solutions: The matchmaker solves:

$$\max_{a,m} \quad \log(1 + a + m) - u(a) - v(m)$$

$$\Rightarrow u'(a) = \frac{1}{1 + a + (1 - a)} = \frac{1}{2}.$$

$$v'(m) = \frac{1}{1 + a + m} = \frac{1}{2}.$$

      The slopes are both $1/2$ (which did not require outside options).

4) Student $i \in \{1, 3, \ldots, 9\}$ is a potential notebook buyer, with a valuation of i for each of up to i notebooks. Similarly, student $j \in \{2, 4, \ldots, 10\}$ is a potential notebook seller, with an opportunity cost each of j for selling up to 2 notebooks.

   a) Find all market clearing prices and quantities. Is the price uniquely pinned down? Is the quantity uniquely pinned down?

      Solution:

      For price $p = 9 + \epsilon$ for small $\epsilon > 0$, the demand is 0 (no buyers value the notebook above its price) and supply is $2 + 2 + 2 + 2 = 8$. For price $p = 9 - \epsilon$, the demand is 9 (from buyer 9) and the supply is still 8. Thus, we must have $p = 9$. At this price, buyer 9 is indifferent and so demand can be $Q^d = \{0, 1, 2, \ldots, 9\}$. There are no indifferent sellers and so supply is $Q^s = 8$, so supply must be 8 to clear the market.

b) What is the largest tax that can be imposed with no deadweight loss?

Solution: For this market to clear without deadweight loss, student 9 must be indifferent. This means that the amount paid by the buyer, $p+t=9$*. For the 8th trade to go through, the net-of-tax amount received by the seller,* p*, must be* $p \geq 8$*. Thus, the largest tax that can be imposed with no deadweight loss is* $t=1$*.*

5) Bonus: Find the efficient matching in the roommate problem with money transfers.

Assume intern I1 can each stay for free with his parents, as can intern I4, and that option is worth payoff 5 to I1 and 1 to I4. On the other hand, interns I2 and I3 have no such good options, and so renting solo is worth zero. What are the transfers among roommates consistent with competitive equilibrium?

Solution: The efficient matching is (I2,I3), (I1,I4), by inspection.

Free matchmaker entry and exit implies the next equalities and inequalities:

$$\begin{aligned} w_1 + w_2 &\geq 7 \\ w_1 + w_3 &\geq 7 \\ w_2 + w_3 &= 7 \\ w_1 + w_4 &= 6 \\ w_2 + w_4 &\geq 5 \\ w_3 + w_4 &\geq 4. \end{aligned}$$

We know that I1 and I4 receive transfers of $w_1 \geq 5, w_4 \geq 1$*. So when I1 and I4 match, the constraints bind:* $w_1 = 5, w_4 = 1$*. This means that I1 transfers* $2 - w_1 = 1$ *to I4, to ensure these payoffs. Finally,* w_2, w_3 *obey respective inequalities:*

$$\begin{aligned} w_2 &\geq 2 \\ w_3 &\geq 2 \\ w_2 + w_3 &= 7 \\ w_2 &\geq 4 \\ w_3 &\geq 3. \end{aligned}$$

Hence, $w_2 = 4, w_3 = 3$*. So it must be the case that I2 and I3 match, but neither transfers anything to the other, since their payoffs are 4 and 3, respectively.*

Part III: Partial Equilibrium (20 points)

6) Consumers with ice cream taste preference $\gamma \geq 1$ are distributed with linear density $f(\gamma) = 1$. There is a fixed health cost $c > 0$ to consuming ice cream. Consumer γ derives utility

$$u(x|p,\gamma) = \frac{1}{\gamma}\sqrt{x} - px - c$$

from x units of ice cream at price p. Consumers can also decide to buy no ice cream, which yields zero utility. What is the demand curve for ice cream?

Solution: For a consumer eating ice cream, the optimal amount to eat is given by the first order conditions: $x^*(\gamma, p) = \left(\frac{1}{2\gamma p}\right)^2$*. They are better off eating ice cream so long as* $\frac{1}{2\gamma^2 p} - \frac{1}{4\gamma^2 p} - c \geq 0 \Rightarrow \gamma \leq \sqrt{\frac{1}{4cp}}$*. This implies we will only have positive demand if the upper bound on* γ *implied by the extensive margin* $\sqrt{\frac{1}{4cp}}$ *is strictly above the lower bound of* γ*, 1. Thus, we will only have positive demand for* $p < \frac{1}{4c}$*. We will have zero demand for higher values of* p*.*

For $p < \frac{1}{4c}$*, demand is*

$$\begin{aligned} D(p) &= \int_1^{\sqrt{\frac{1}{4cp}}} \left(\frac{1}{2\gamma p}\right)^2 d\gamma \\ &= \frac{1}{4p^2} \int_1^{\sqrt{\frac{1}{4cp}}} \gamma^{-2} d\gamma \\ &= -\frac{1}{4p^2}(\sqrt{4cp} - 1) \\ &= \frac{1 - 2\sqrt{cp}}{4p^2}. \end{aligned}$$

Therefore, demand is $Q^d(p) = \begin{cases} 0, & \text{if } p \geq \frac{1}{4c} \\ \frac{1-2\sqrt{cp}}{4p^2}, & \text{otherwise.} \end{cases}$