Economics 713: Midterm

February 13, 2020

Cite any theorems you apply. Rigorously justify everything.
You have 75 minutes for 75 points (+15 Bonus). May the Force be with you.

1. thank u, next **[20pts]**

Consider the following matrix of payoffs:

	Sean	Ricky	Pete	Malcolm
Ari	4,1	3,2	2,3	1,4
Bibi	3,3	4,1	2,4	1,3
Cali	2,4	3,3	4,1	1,2
Dani	4,2	2,4	3,2	1,1

(a) Find the stable matchings when women propose and when men propose. [8]

Women Propose
(1) $A \to S$, $B \iff R$, $C \iff P$, $D \iff S$
(2) $A \iff R$
(3) $B \iff S$
(4) $D \iff P$
(5) $C \iff R$
(6) $A \iff P$
(7) $D \iff R$
(8) $C \iff S$
(9) $B \iff P$
(10) $A \iff M$

Men Propose
(1) All men propose to their highest woman. Matches are set immediately, with: $A \iff M$, $B \iff P$, $C \iff S$, $D \iff R$.

In both cases, the match is identical: $\{(A, M),\ (B, P),\ (C, S),\ (D, R)\}$.

(b) Add a superstar woman x^* and a man y^* to an n-man n-women matching model, with x^* strictly most preferred by all men, and y^* strictly most preferred by all women. What happens to the two DAA matchings and the set of stable matchings? [3]

In the first round all men (women) propose to the 'superstar' woman (man). The superstar woman (man) will accept the proposal of the superstar man (woman). The remainder of proposals occur in exactly the same order as the original problem, with the resulting matches between non-superstars exactly the same as before.

In a two-sided matching market with n men and n women, the maximum number of steps in the DAA is $n^2 - 2n + 2$. Assume strict preferences and all agents want to match. For each statements, prove that the statement is true, or provide a counterexample.

(c) True or False: When one side proposing leads to the maximum number of DAA rounds, the DAA ends in a single round when the other side proposes. [2]

(Hint: What happens when you change Malcolm's payoffs in part (a)?)

False. In the above example, if Malcolm prefers $D \succ C \succ B \succ A$, then the algorithm stops in n rounds.

(d) True or False: Whenever preferences are induced by comonotone utility functions, the DAA takes the maximum number of rounds. [2]

False. Though this is true when $n = 2$ (since both men propose to the same woman in the first round, leaving one to search in the second round), with $n > 2$ men, comonotone payoffs leads all men to propose to the same woman in the first round; all but one are rejected; then all remaining men propose to the same remaining woman in the second round, etc. This takes n rounds.

(e) (Bonus) True or False: If the DAA takes the maximum number of rounds to conclude, then the matching produced by the DAA is pessimal for those proposing. [5]

True. For the side receiving proposals, all but one (at most) receive their optimal partner (because they each reject $n - 1$ proposals, so must end up with their best partner). Clearly this is optimal for those receiving proposals, so must be pessimal for those making proposals. Thus the matching is unique.

2. New Salop State of Mind **[25pts]**

The city of New Salop is built on the interior of the unit disk in the xy-plane, centered at the origin (i.e., $x^2 + y^2 \leq 1$). Residents (each with zero mass, i.e. each negligible) are indexed by their locations (x, y) distributed uniformly with density 1.[1] Residents can work for a unit mass of firms, indexed by z distributed uniformly on $[0, 1]$. When resident (x, y) matches with firm z, match payoffs to resident and firm are respectively:

$$f(z|x, y) = 2 - 2z + 2z\sqrt{x^2 + y^2}, \qquad \text{and} \qquad g(x, y|z) = 2z - z\sqrt{x^2 + y^2}$$

Workers and firms both have an outside option of 0.

(a) Does everyone wish to match in the NTU model? [2]

Yes. All potential matches have utility weakly greater than zero for both residents and firms.

(b) Hereafter, assume transferable utility. Describe all stable matchings.[2] [3]

(Hint: The analysis is neater if you summarize residents by a scalar index.)

The first step in this problem is realizing that in every expression, x and y always appear as $\sqrt{x^2 + y^2}$, which we can just write as r — i.e. the distance from the origin.

[1] This means that for any set of residents A, the mass of those residents is equal to the area of A.

[2] Note: We originally meant for this problem to be asking about the NTU case, but this was not clear from the wording of the question. The solution below gives the solution for NTU, but it was sufficient for full points EITHER to solve the NTU case OR to realize that in the TU world, the efficient matching is uniquely stable, and so the answers to (b) and (c) coincide. Apologies for any confusion.

With NTU, residents and firms prefer the lowest indexed members of the opposite side of the market, and all prefer to match since utilities are always positive. So matches are PAM with the lowest indexed residents matching, and the highest indexed residents remaining unmatched.

(c) Describe the efficient matching when utility is transferable. [5]

Joint surplus is:

$$h(r, z) = 2 + rz$$

The joint payoff function is SPM, and the efficient match is PAM!, but this time, the increasing joint surplus function means we need the highest indexed residents matching, and the lowest indexed residents remaining unmatched.

(d) Is there a short side of the market? What does this imply about both sides' wages? [5]

Firms are on the short side of the market, since the mass of firms is 1 while the mass of residents is π.

This implies that wages are uniquely pinned down. We know that the lowest surplus-producing match is that between $z = 0$ and the lowest-matching resident (who will have a positive index). At this point, wages must add up to two (the value of joint surplus). Furthermore, the outside options require that wages be weakly positive.

We claim that wages of the lowest-matching firm and worker is uniquely pinned down at $v_0 = 0$ and $w_0 = 2$. To see this, suppose that the lowest worker r_0 instead earns a positive wage $v_0 > 0$, and so firm 0 gets $w_0 = 2 - v_0 < 2$ by zero profits. But then a matchmaker could enter and offer to pay a slightly lower worker $r' < r_0$ to match with firm 0. Such a worker is strictly willing to match for any wage $\varepsilon > 0$ (it beats not matching!), and for small enough $\varepsilon > 0$, and r' close enough to r_0, the match maker can afford to pay firm 0 more than $2 - v_0$.

(e) (Bonus) Write the wage equations that decentralize the efficient matching. [10]
(Hint: Find workers' wages, and deduce firms' wages by zero matchmaker profits.)

1. Matches: Because this is a circular city and agents are distributed uniformly, the total mass of residents with $\sqrt{x^2 + y^2} \leq r$ is the area of the circle with radius r. Normalizing gives the cumulative mass function $F(r) = \pi r^2$. Given this cmf, positive assortative matching, and the fact that we want the highest residents matching, we know matches will form according to

$$\pi - F(r) = 1 - G(z) \implies \pi - \pi r^2 = 1 - z$$

This implies that worker $r(z)$ matches with firm z, and equivalently firm $z(r)$ matches with worker r, where:

$$r(z) = \sqrt{\frac{\pi - 1 + z}{\pi}}, \qquad \text{and} \qquad z(r) = \pi(r^2 - 1) + 1$$

The lowest resident r_0 is characterized by:

$$\pi - \pi r_0^2 = 1 \implies r_0 = \sqrt{\frac{\pi - 1}{\pi}}$$

2. Differentiating: Differentiating the match payoff function gives:

$$\frac{\partial h}{\partial r}(r,z) = z, \qquad \text{and} \qquad \frac{\partial h}{\partial z}(r,z) = r$$

3. Integrating: Thus, wages for workers are given by:

$$v(\bar{r}) = \int_{r_0}^{\bar{r}} \frac{\partial h}{\partial r}(r, z(r))dr = 0 + \int_{\sqrt{(\pi-1)/\pi}}^{\bar{r}} \left[\pi(r^2-1)+1\right] dr$$

$$= (1-\pi)\left(\bar{r} - \sqrt{\frac{\pi-1}{\pi}}\right) + \frac{\pi}{3}\left(\bar{r}^3 - \sqrt[3/2]{\frac{\pi-1}{\pi}}\right)$$

For firms, we know by the zero matchmaker profit condition that $w(\bar{z}) = h(r(\bar{z}),\bar{z}) - v(r(\bar{z}))$.[3]

3. Heterogeneous Firms **[25pts]**

An economy has a continuum of small potential firms indexed by independent parameters $x \in [0,\infty)$ and $y \in [1,2]$. Potential firm (x,y) can produce quantity $q \geq 0$ at a cost:

$$c(q|x,y) = x + yq^2,$$

where x is a fixed cost, escapable in the long-run. Potential firms (x,y) have a unit density on $[0,\infty) \times [1,2]$. Thus, the mass with $x \leq \bar{x}$ is $\bar{x}$, and the joint cmf is $F(x,y) = x(y-1)$.

(a) What is the short run supply curve of firm (x,y)? What is its long-run supply? [5]

Any firm that produces will choose quantity to equate marginal cost and price:

$$p = MC(q) = 2yq \implies q^S(p|x,y) = p/2y$$

This is the short run supply of firm (x,y). This doesn't depend on x, since x is sunk for all firms in the market.

In the long run, firm (x,y) only produces if it can make positive profits:

$$x + p^2/4y \leq p^2/2y \implies x \leq p^2/4y$$

So long-run supply is:

$$q_S^{LR}(p|x,y) = \begin{cases} p/2y & \text{if} \quad x \leq p^2/4y \\ 0 & \text{otherwise} \end{cases}$$

[3] Aside: If you solve were to solve this explicitly (not required), you would get

$$w(\bar{z}) = 2 + \bar{z}r\sqrt{\frac{\pi-1+\bar{z}}{\pi}} + (\pi-1)\left(\sqrt{\frac{\pi-1+\bar{z}}{\pi}} - \sqrt{\frac{\pi-1}{\pi}}\right) - \frac{\pi}{3}\left(\sqrt[3/2]{\frac{\pi-1+\bar{z}}{\pi}} - \sqrt[3/2]{\frac{\pi-1}{\pi}}\right)$$

(b) Derive the long-run supply curve. [3]

For every marginal cost $y \in [0, 1]$ of a firm, the lowest fixed cost firms produce (i.e. those with $x < p^2/4y$). Thus we can write:

$$Q_S^{LR}(p) = \int_{y=1}^{2} \int_{x=0}^{p^2/4y} p/2y \; dx \; dy = \int_1^2 \frac{p^3}{8y^2} dy = -\frac{p^3}{8y} \Big|_1^2 = p^3/16$$

(c) Calculate the short-run supply curve starting at a long run equilibrium price $\bar{p} > 0$. [5]

Starting at $\bar{p}$, we know that firms (x, y) with $x \leq \bar{x} = \bar{p}^2/4y$ are in the market. Since marginal cost is increasing from 0, no one exits in the short run, and supply reflects just an intensive margin re-optimization:

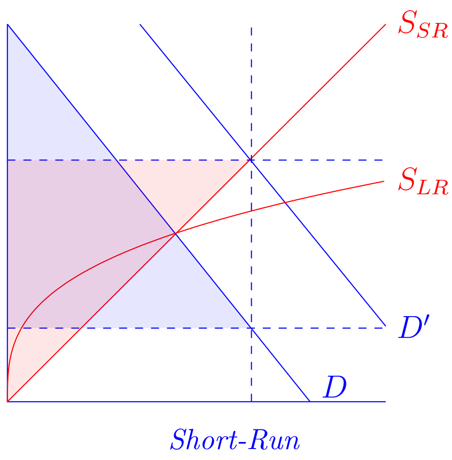
$$\begin{aligned} Q_S^{SR}(p|\bar{p}) &= \int_{y=1}^{2} \int_{x=0}^{\bar{p}^2/4y} (p/2y) dx \; dy \\ &= \int_{y=1}^{2} \int_{x=0}^{\bar{p}^2/4y} (p/2y) dx \; dy \\ &= \int_{y=1}^{2} \left(\frac{p \cdot \bar{p}^2}{8y^2} \right) \; dy \\ &= -\frac{p\bar{p}^2}{8y} \Big|_1^2 \\ &= p\bar{p}^2/16 \end{aligned}$$

(d) Suppose the original long-run equilibrium price $\bar{p}$ arose from market demand $P(Q) = a - Q$. Assume that demand shifts to $P(Q) = A - Q$, where $A > a > 0$. *Qualitatively* compare the short-run and long-run equilibrium price and quantity changes. [5]

In the long run, the new demand curve intersects long-run supply at a higher price and quantity, so the market can support a larger number of firms. In the short run, however, the increased demand must be met entirely by firms willing to supply at the level of demand. Thus the short run supply curve will be everywhere steeper than long run supply, meaning that the intersection point is at a price and quantity higher than the original price and quantity, but this quantity is lower than the new long run equilibrium quantity, and the price is higher than the long run equilibrium price. (See picture below)

(e) Suppose that the upward demand shift in part (d) happened because the government started a small unit subsidy $\sigma > 0$. Do suppliers capture a greater share of the subsidy in the long-run or the short-run? *Graphically* illustrate your claim only. [7]

Graphically, we can see that suppliers capture a larger share of the subsidy in the short run, as their supply is more elastic in the long run:



Aside: Mathematically, tax-burdens/subsidy-shares for supply (given a small tax/subsidy) are given by:

$$SR: \frac{|\varepsilon_{SR}|}{|\eta_{SR}| + |\varepsilon_{SR}|}, \qquad \text{and} \qquad LR: \frac{|\varepsilon_{LR}|}{|\eta_{LR}| + |\varepsilon_{LR}|}$$

where ε is demand elasticity and η is supply elasticity. And we know

$$|\varepsilon_{SR}| \cdot |\eta_{LR}| \geq |\varepsilon_{LR}| \cdot |\eta_{SR}|$$

All told, suppliers capture a larger share of the subsidy in the short run.

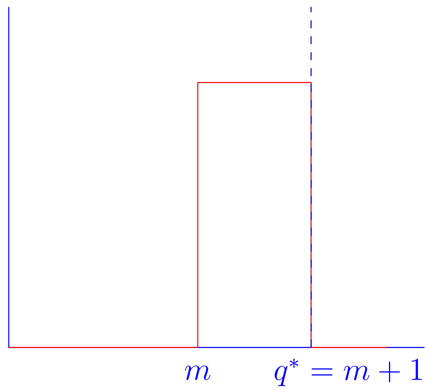
4. Zuck & Donny **[20pts]**

Mark Zuckerberg owns everyone's data and wishes to sell it to advertisers who are indexed by $m \in [0, \infty)$ with a density of 1. The marginal revenue to advertiser m of receiving quantity q of data is 1 iff $m \leq q \leq m+1$, and otherwise 0:

$$MR(q|m) = \mathbb{I}_{m \leq q \leq m+1}$$

Zuck's friend Donny levies a per-unit data tax of $\tau > 0$. As Zuck finds it burdensome to charge per unit of data, he charges advertisers a flat access fee φ regardless of quantity.

(a) Plot the marginal revenue function of advertiser m. If advertiser m buys data at tax τ, what is his optimal quantity, and what is the total profit at this quantity? [6]



If an advertiser buys quantity q, then it is necessarily the case that $\tau < 1 = MR(q)$. The profit maximizing quantity is thus $q(m) = m + 1$, where the total profit is 1.

(b) Derive the aggregate demand function for data as a function of φ and τ. [8]

Advertisers require that total revenues exceed total costs, and thus buy only when:

$$\tau(m+1) + \varphi \leq 1 \implies m \leq \frac{1-\tau-\varphi}{\tau}$$

Thus, total demand is given by:

$$\begin{aligned} Q_D(\tau, \varphi) &= \int_0^{\frac{1-\tau-\varphi}{\tau}} (m+1)\, dm \\ &= \frac{1}{2}m^2 + m \Big|_0^{\frac{1-\tau-\varphi}{\tau}} \\ &= \frac{1-\tau^2+\varphi^2-2\varphi}{2\tau^2} \end{aligned}$$

(c) For any tax $\tau > 0$, what is Zuck's profit maximizing entrance fee? [6]

Since Zuck secures fee φ for each advertiser who buys, his profits are:

$$\pi(\varphi) = \varphi \int_0^{\frac{1-\varphi-\tau}{\tau}} dm = \frac{\varphi - \varphi\tau - \varphi^2}{\tau}$$

As this is concave in φ, the optimal fee must (by the FOC) satisfy:

$$\frac{1-\tau}{\tau} = \frac{2\varphi}{\tau} \implies \varphi^*(\tau) = \frac{1-\tau}{2}$$

Intuitively, the tax τ deters some advertisers from buying. Of those who remain, the optimal fee set by Zuckerberg deters precisely half — where the marginal gain to charging a higher fee precisely balances out the marginal loss of losing advertisers.