Econ 713 Midterm

UW-Madison

Valentine's Day ♡, 2023 in class

♡ *There are* 75 ***points in 75 minutes.*** *Points are at the right.*
♡ **Rigorously justify everything with graphs or algebra or a known theorem.** *Enjoy!*

1. ♡ The payoff for matches between women (x) and men (y) are given by the table below,

	x_1	x_2	x_3
y_1	1,3	2,2	3,1
y_2	3,1	1,3	2,2
y_3	2,2	3,1	1,3

(a) Find all stable matchings when payoffs are non-transferable, and utility is linear on payoffs **[10 Pts]**

(b) Are any of the stable matching found in (a) socially efficient? Is any feasible matching inefficient? Explain. **[5 Pts]**

Solution:
(a) We run the DAA to check for stable matches. We start by y proposing,

$$y_1 \to x_3,\ y_2 \to x_1,\ y_3 \to x_2,\ \mathcal{M}_y = \{(1,3),(2,1),(3,2)\}$$

The algorithm ends in one step. Now, we check if it is unique by letting x propose matches in the DAA:

$$x_1 \to y_1,\ x_2 \to y_2,\ x_3 \to y_3,\ \mathcal{M}_x = \{(1,1),(2,2),(3,3)\}$$

Once more the algorithm ends in one step. Since $\mathcal{M}_y \neq \mathcal{M}_x$ we check if there is any other stable match involving the unused pairs $\{(1,2),(2,3),(3,1)\}$. It's immediate that the allocation $\mathcal{M} = \{(1,2),(2,3),(3,1)\}$ has no blocking pair, and so is a third stable matching.

(b) All three allocations found in (a) are socially efficient. This follow from $h(x,y) = 4$ for all x, y, i.e the production function is modular and so any feasible allocation is socially efficient.

2. ♡ Passengers p_1, p_2, and p_3 want to travel to different locations. Uber drivers d_1, d_2, and d_3 in different city locations may provide transportation. Passenger preferences depend on the destinations and not the drivers. Drivers just have preferences over the destinations, reflecting the proximity to their homes. This table illustrates the payoffs (assume utility equals payoff):

	d_1	d_2	d_3
p_1	5,1	5,2	5,3
p_2	3,3	3,4	3,2
p_3	1,2	1,4	1,1

Passengers who remain unmatched have payoff $-\beta < 0$, of taking the *bus*. Since unmatched drivers can wait for *new* passengers, they have an outside option $n > 0$.

(a) Suppose all passengers match with a driver in a competitive equilibrium. Who matches with whom in this equilibrium? **[3 Pts]**

(b) Let payoffs be transferable. Let v_i and w_j be the payoffs of passenger i and driver j. What conditions on (v_i) and (w_j) ensure that the matching in (a) is a competitive equilibrium? How much higher is passenger 2's payoff than passenger 3's? **[7 Pts]**

(c) Start with the table payoffs, but also allow transfers: passenger $i = 1, 2, 3$ pays his driver a fare $f_i \geq 0$. For which fares is the matching in (a) a competitive equilibrium? **[5 Pts]**

(d) Let $\beta = 1$ and $n = 7$. Who matches with whom in this situation? **[10 Pts]**

Solution:

(a) Since passengers payoffs are independent of drivers we only need to maximize the allocation of drivers. Thus, the matches are $\{(1,3),(2,1),(3,2)\}$.

(b) For equilibrium, we need

$$\begin{aligned} v_1 + w_1 &\geq 6, & v_1 + w_2 &\geq 7, & v_1 + w_3 &= 8 \\ v_2 + w_1 &= 6, & v_2 + w_2 &\geq 7, & v_2 + w_3 &\geq 5 \\ v_3 + w_1 &\geq 3, & v_3 + w_2 &= 5, & v_3 + w_3 &\geq 2 \end{aligned}$$

$$v_i \geq -\beta, \quad w_j \geq n, \quad \text{for all } i, j \in \{1,2,3\}$$

Next we use that $w_1 = 6 - v_2$ and $w_2 = 5 - v_3$ to obtain the desire bounds,

$$v_2 - v_3 \leq 3, \quad v_2 - v_3 \geq 2$$

Thus, the payoff of passenger 2 is at least two units higher than passenger three, but not greater than 3.

(c) First we comment that having this fare is like the house price in Shapley and Shubik (1971) in the sense that it is not the shadow value, but can be deduced knowing the multiplier. Consider the case of the pair $(1,3)$. The fare f_1 follow $v_1 = 5 - f_1$ and $w_3 = 3 + f_1$, replacing this over the outside option constrain for passengers,

$$5 - f_1 \geq -\beta, \quad 3 - f_2 \geq -\beta, \quad 1 - f_3 \geq -\beta$$

for the driver side,

$$3 + f_1 \geq n, \quad 3 + f_2 \geq n, \quad 4 + f_3 \geq n$$

Altogether,

$$f_1 \in [n-3, 5+\beta], \quad f_2 \in [n-3, 3+\beta], \quad f_3 \in [n-4, 1+\beta]$$

(d) By replacing the parameters we notice that there is no feasible fare for passenger 3 since $f_3 \in [3, \infty) \cap (-\infty, 2] = \emptyset$. Hence, there is no equilibrium match in which passenger 3 has a driver. This free driver 2 to be re-allocated. Note that the outside option of each driver are the same, and so we can swap driver 1 that is taking passenger 2 for driver 2 to improve the total surplus of the allocation. Is immediate to see that there exist a feasible fare for $(2, 2)$ and so the efficient matching now is $\{(1, 3), (2, 2)\}$, with d_1 and p_3 unmatched.

3. ♡ Consider a two sided market of Researchers and Assistants. Researchers are indexed by their ideas $x \in [0,1]$, with mass cdf $F(x) = x^2$. Assistants are indexed by their coding skill $y \in [0,1]$, with mass cdf $G(y) = y$. The publication output of researcher x and assistant y is

$$h(x,y) = (x+y-1)^2$$

The outside option for unmatched researchers and assistants is zero.

(a) Find the efficient matching of this situation. Who matches with whom? **[10 Pts]**

(b) Obtain the competitive equilibrium wages $w(x)$ and $v(y)$. **[10 Pts]**

(c) **[Bonus]** Let the social planner impose a fixed tax $t > 0$ on match output. How does this impact the competitive equilibrium? Clearly identify the effect at each side of the market. **[10 Pts]**

Hint: *Think about the lowest payoff match.*

Solution:

(a) First we observe that payoffs $h(x,y)$ are supermodular: $h_{xy} = 2 > 0$. Hence, PAM is efficient, and so $1 - F(x) = 1 - G(y(x)) \Rightarrow x^2 = y(x)$. So the matching is (z, z^2) for $0 \leq z \leq 1$.

(b) We set up the middleman's problem $\max_{x,y} \pi(x,y)$, where

$$\pi(x,y) \equiv (x+y-1)^2 - w(x) - v(y)$$

and we evaluate the FOC at the efficient (PAM) matching:

$$\frac{\partial \pi(x,y)}{\partial x} = 2(x+y-1) - w'(x) = 0\big|_{(x,y)=(z,z^2)} \qquad w'(z) = 2(z + z^2 - 1)$$

$$\frac{\partial \pi(x,y)}{\partial y} = 2(x+y-1) - v'(y) = 0\big|_{(x,y)=(\sqrt{z},z)} \qquad v'(z) = 2(z^{1/2} + z - 1)$$

Integrating,

$$w(z) = z^2 + \frac{2}{3}z^3 - 2z + C_x \quad \text{and} \quad v(z) = \frac{4}{3}z^{3/2} + z^2 - 2z + C_y \tag{$\clubsuit$}$$

Next we use the middleman zero profits condition to obtain the constants of integration, since we know that $(0,0)$ is a PAM match:

$$h(0,0) - w(0) - v(0) = 0 \implies 1 = C_x + C_y$$

In addition, since each side has an outside option of zero $C_x, C_y \geq 0$. Furthermore, the worst match must also be willing to participate, i.e. the wage for the worst match must no less than zero as well. We can characterize the worst match by the type of researcher $\tilde{z} \in \text{argmin } h(z, z^2)$. Thus, the constant C_x and C_y are fully characterize by $w(\tilde{z}) \geq 0$ and $C_y = 1 - C_x$, where $C_x, C_y \geq 0$.

(c) Note that the worst match is an interior point $\tilde{z} \in (0,1)$, and output falls on $[0, \tilde{z}]$ and increases on $[\tilde{z}, 1]$. Intuitively, matches near $\tilde{z}$ are no longer profitable after the tax. We will find two zero profit cutoff points for any given tax.

$$h(z, z^2) = t \quad \Rightarrow \quad z + z^2 - 1 = \pm\sqrt{t}$$

Thus, we can express the cutoff points for researchers as:

$$\overline{z} = \frac{-1 + \sqrt{1 + 4(1 + \sqrt{t})}}{2} \quad \text{and} \quad \underline{z} = \frac{-1 + \sqrt{1 + 4(1 - \sqrt{t})}}{2}$$

Finally, the new competitive equilibrium is PAM with (z, z^2) where $z \in [0, \underline{z}] \cup [\overline{z}, 1]$. This means that researchers $x \in [\underline{z}, \overline{z}]$ and assistants $y \in [\underline{z}^2, \overline{z}^2]$ are unmatched.

4. ♡ Hogwarts Legacy is the newest Harry Potter video game. Consumers care about: (1) their *taste* θ for the game, (2) how much *time* t they have to play, and (3) its price p. Specifically, their utility from purchase is:

$$\theta t - p$$

They get zero utility if they do not buy. Assume that the time each consumer has to play $t \in [0, 1]$ has mass cdf $F(t) = t^{\beta}$, with $\beta > 2$, and is independent of the consumer's taste parameter $\theta \sim U[0, 1]$. Assume a mass M of potential consumers.

Find the aggregate demand function for Hogwarts Legacy. **[15 Pts]**

Hint 1: The mass of consumers with parameters θ and t in any set $S \subset \{(\theta, t)|0 \leq \theta, t \leq 1\}$ is the integral $\int MF'(t)d\theta dt$ over set S.

Hint 2: Be careful when stating the integration limits. What is demand when $p > 1$?

Solution: A consumer buys Hogwarts Legacy if and only if $\theta t \geq p$. If $p > t$, then no consumer buys the game since $\theta \leq 1$. For any fixed $t \geq p$, the mass of consumer tastes θ that lead to a purchase is:

$$\int_{p/t}^{1} M d\theta = M\left(1 - \frac{p}{t}\right)$$

Integrating across times t in $(p, 1)$, the mass of consumers buying is:

$$\int_{p}^{1} M\left(1 - \frac{p}{t}\right)\beta t^{\beta-1}dt = M\left[t^{\beta} - p\frac{\beta t^{\beta-1}}{\beta - 1}\right]\Bigg|_{p}^{1} = M\left[1 - p^{\beta} + \beta\frac{p^{\beta} - p}{\beta - 1}\right]$$

Altogether, aggregate demand function is

$$D(p) = \begin{cases} 0 & \text{if } p > 1 \\ \\ M\left[1 + \dfrac{p^{\beta} - \beta p}{\beta - 1}\right] & \text{if } p \leq 1 \end{cases}$$