Addendum to Compiani, Magnolfi and Smith (2024)

Risk Neutral Supply: Proof of Theorem 0

We derive the earlier formula directly, considering all the possible integer ticket winners — highlighting the positive probability of a shared jackpot.

Theorem 0 (Risk Neutrality) *The inverse supply for a rollover lottery is:*

$$L(Q|J) = p - w - [J/Q + p(1-\tau)][1 - e^{-\alpha Q}].$$

Here, we assume Q is a natural number. The chance of exactly k *other winners* is $C(Q-1,k)\pi^k(1-\pi)^{Q-1-k}$, where $C(n,k) = \frac{n!}{k!(n-k)!}$. So the expected winnings are

$$w + \pi[J + p(1-\tau)Q] \sum_{k=0}^{Q-1} \frac{C(Q-1,k)}{k+1} \pi^k (1-\pi)^{Q-1-k} \tag{25}$$

So winners secure a $1/(k+1)$ share when k others pick the same winning sequence. We use generating functions to collapse this to (1). Let $\rho \equiv \pi/(1-\pi)$, and define a function $f(\rho) \equiv \sum_{k=0}^{Q-1} \frac{C(Q-1,k)}{k+1} \rho^{k+1}$. Then expected winnings (25) are:

$$w + [J + p(1-\tau)Q] f(\pi/(1-\pi))(1-\pi)^Q$$

Since $f'(\rho) \equiv (1+\rho)^{Q-1}$ and $f(0) = 0$, integration yields $f(\rho) = \frac{1}{Q}[(1+\rho)^Q - 1]$. As $1 + \rho = 1/(1-\pi)$, and $(1-\pi)^Q = e^{-\alpha Q}$, winnings per ticket (25) imply losses:

$$w + [J + p(1-\tau)Q] \cdot \frac{(1/(1-\pi))^Q - 1}{Q} \cdot (1-\pi)^Q$$

This reduces to the expected gains subtracted in (1) from the price p. $\square$