Economic Theory 713A

Economics of Markets

Lones Smith

March 3, 2025

Wisconsin

General Equilibrium and the Gold Rush

- *Partial equilibrium*: one-market world, often with quasi-linear utility — where "money" subsumes all other goods
- *General equilibrium* multi-market world: Markets interact!
  - Eg. Arrow missing markets; quantity constraint token markets
- Sam Brannan
  - Richest man in California after Gold Rush of 1849
  - "Gold! Gold on the American River!" - Sutter's Mill, California
  - Brennan owned only store between San Francisco & gold fields
  - Sold 20 cent pans for $15 each

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CALIFORNIAN

BY B. R. BUCKELEW.

SAN FRANCISCO, MARCH 15, 1848.

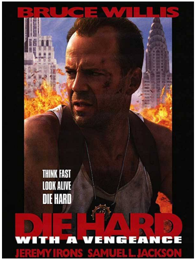
GOLD MINE FOUND.—In the newly made raceway of the Saw Mill recently erected by Captain Sutter, on the Ameri-

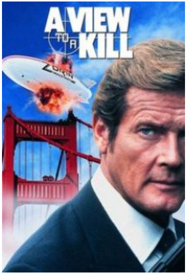
California Gold Rush Map


WHEN EVERYONE DIGS FOR GOLD
SHOVELS $99
NVIDIA
Microsoft
Meta
SELL SHOVELS

General Equilibrium in the Movies

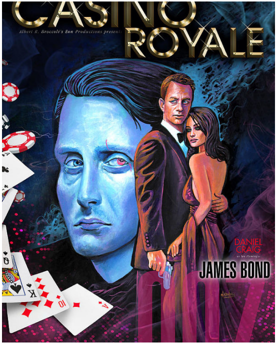
- Goldfinger: evil mastermind tried to irradiate Fort Knox gold $\Rightarrow$ his own gold would $\uparrow$ in value
- Die Hard with a Vengeance: same plan for the gold in NY Fed.
- A View to a Kill: bad guy wants to trigger earthquake to destroy Silicon Valley, and then monopolize microchip market.

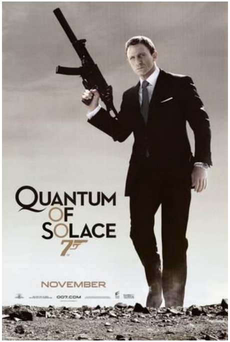




General Equilibrium in the Movies

- Casino Royale: bad guy shorts airline stocks, while planning to destroy a luxury jetliner on its maiden voyage.
- Quantum of Solace: bad guy wants to dam Bolivia's fresh water supply to create a Bolivian water monopoly (total joke).





Applied General Equilibrium: Zoom

- Stock price of Zoom Video Communications (NASDAQ: ZM)

Applied GE: Rheinmetall Artillery Company

- Russia invaded Ukraine on February 24, 2022
- Two biggest artillery firms: General Dynamics and Rheinmetall

General Equilibrium Model of an Exchange Economy

- *Exchange economy* $\mathcal{E} = (\{u^i\}, \bar{\mathbf{x}})$.
  - $L \geq 2$ goods $\ell \in \{1, 2, \ldots, L\}$
  - $n \geq 2$ traders $i \in \{1, 2, \ldots, n\}$
  - Consumer i has *endowment* $\bar{\mathbf{x}}^i = (\bar{x}_1^i, \bar{x}_2^i, \ldots, \bar{x}_L^i)' \in \mathbb{R}_+^L$
  - A goods *allocation* is a matrix $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n) \in \mathbb{R}_+^{nL}$.
  - Trader i has utility $u^i : \mathbb{R}_+^L \to \mathbb{R}$.
  - Trader i's income is the market value $\mathbf{p} \cdot \bar{\mathbf{x}}^i$ of his endowment
  - So every trader solves a traditional consumer theory problem
- Prices $\mathbf{p} = (p_1, p_2, \ldots, p_L) \in \mathbb{R}_+^L$ in some unit of account
  - Jevons (1875): Money is a *store of value*, *unit of account*, and *medium of exchange*, *standard of deferred payment*
  - Here, it is only a unit of account, and so $\exists$ degree of freedom.
  - Each trader sells his endowment to the market, valued at the unit of account prices, and then buys his optimal bundle.
  - We assume that all transactions realize by time-0 contracts
  - Modern financial transactions, together with bankruptcy laws, violate this idyllic world (hence the 2008 Financial Crisis)

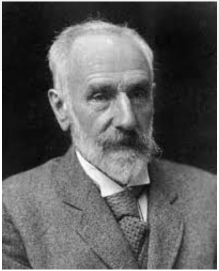
General Equilibrium

- A trader's wealth is the market value of his endowment
- *Budget set* $\mathcal{B}^i(\bar{\mathbf{x}}^i, \mathbf{p}) = \{\mathbf{x}^i \in \mathbb{R}^L_+ | \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \bar{\mathbf{x}}^i\}$
- Traders optimize, given prices: Trader $i = 1, 2 \ldots, n$ solves:

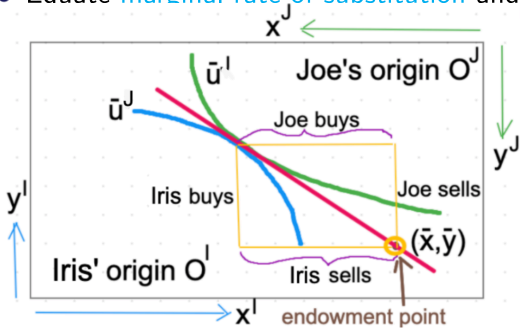
$$\max u^i(\mathbf{x}^i) \ \text{ s.t. } \mathbf{x}^i \in \mathcal{B}^i(\bar{\mathbf{x}}^i, \mathbf{p})$$

- Allocation $\mathbf{x} \in \mathbb{R}^{nL}_+$ is feasible for $\mathcal{E}$ if $\sum_{i=1}^n x^i_\ell \leq \sum_{i=1}^n \bar{x}^i_\ell \ \forall \ell$
  - free disposal of goods $\Rightarrow$ weak inequality
  - We say that markets clear in this case
- A competitive equilibrium $(\mathbf{x}, \mathbf{p})$ of $\mathcal{E}$ is a feasible (market-clearing) allocation $\mathbf{x}$ s.t. all traders optimize, given $\mathbf{p}$
- Feasible allocation $\mathbf{x}$ is socially optimal if $\nexists$ feasible allocation $\mathbf{z}$ with
  - no one worse off: $u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i)$ for all $i = 1, \ldots, n$,
  - some trader j strictly better off: $u^j(\mathbf{z}^j) > u^j(\mathbf{x}^j)$ for some j
  - An allocation where one trader owns everything is efficient.

Edgeworth Boxes for $n = 2$ Traders



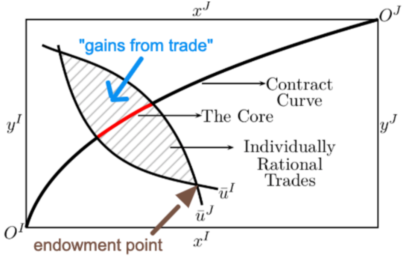
- Francis Ysidro Edgeworth
  - *Mathematical Psychics* (1881)
  - introduced indifference curves
  - founding editor: *Economic Journal*
- Trader Iris and Trader Joe trade goods x, y from endowment to an optimal allocation
- Assume an interior solution with smooth preferences.
- Equate marginal rate of substitution and price ratio: $\frac{u_x}{u_y} = \frac{p_x}{p_y}$



Competitive Equilibrium and Social Efficiency

- Individually rational (IR) allocation $\mathbf{x}$ obeys $u^i(\mathbf{x}^i) \geq u^i(\bar{\mathbf{x}}^i)\ \forall i$
  - No trading mechanism, even with market power, can violate the IR constraint — exceptions for the Godfather
  - Trade occurs due different preferences and/or endowments
  - Divergent marginal rates of substitution $\Rightarrow$ gains from trade
- Contract curve: socially efficient allocations (pairwise optimal)
- The core is IR *and* on the contract curve
- A competitive equilibrium for $\mathcal{E}$ is a pair $(\mathbf{x}, \mathbf{p})$ s.t. $\mathbf{x}$ is feasible, and optimal for traders, given prices $\mathbf{p}$ (via budget sets)

$\Rightarrow$ A competitive equilibrium is in the core since $\mathbf{x}^i \in \mathcal{B}^i(\bar{\mathbf{x}}^i, \mathbf{p})\ \forall i$

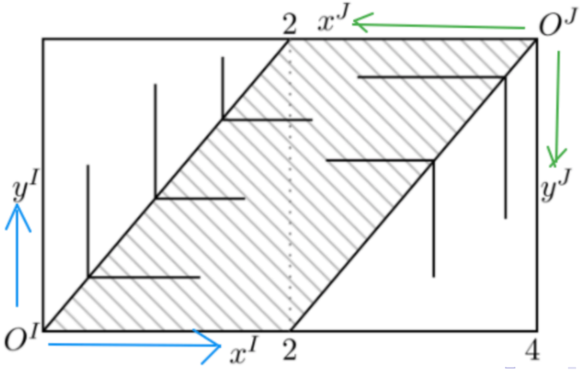


Thinker: Social Efficiency with Perfect Complements

- Utility functions $u^I(x,y) = \min\{x,y\}$ & $u^J(x,y) = \min\{x,y\}$
- Endowments $\bar{x}^I = \bar{x}^J = 2$ and $\bar{y}^I = \bar{y}^J = 1$

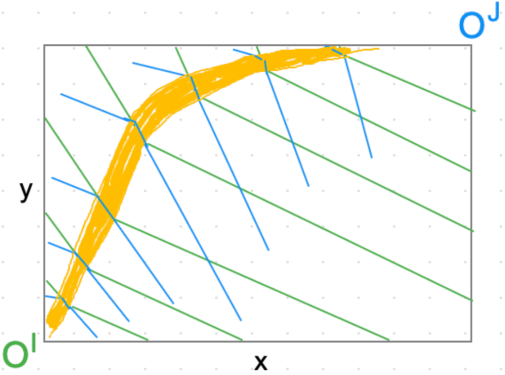
$\Rightarrow$ Contract curve is a region, not a curve, because preferences are not strictly monotone

- Exercise: Show any point is inefficient iff it is non-shaded



Thinker: Social Efficiency with Imperfect Complements

- Increasing preferences that with at least one party strictly convex is needed to ensure a contract curve and not region
- The orange region is socially efficient, given Iris' green and Joe's blue indifference curves (monotone preferences)

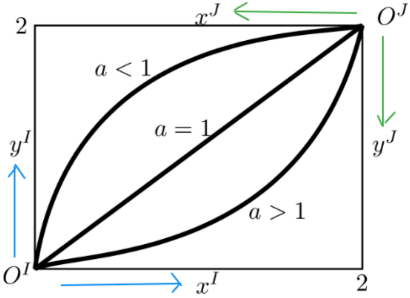


Social Efficiency with Smooth Strictly Convex Preferences

- Cobb-Douglas utility: $u^I(x, y) = x^\alpha y$ and $u^J(x, y) = xy$
- Endowments $\bar{x}^I = \bar{x}^J = \bar{y}^I = \bar{y}^J = 1$.
- Contract curve: $MRS^I_{x,y} = MRS^J_{x,y}$

$$\alpha y^I/x^I = y^J/x^J \Rightarrow \alpha y^I(2 - x^I) = x^I(2 - y^I) \Rightarrow y_I = \frac{2x^I}{\alpha(2 - x^I) + x^I}$$

- Contract curve is above or below the diagonal as $\alpha \lesseqgtr 1$.
- As $\alpha \uparrow$, Iris values good x more, and he efficiently gets more x



Competitive Equilibria are Socially Efficient

- Since trade is win-win, it makes sense that self-interest is good
- Adam Smith (1723–90)
  - 1759: "Theory of Moral Sentiments" explored empathy
  - 1776: "Inquiry into the Nature and Causes of the Wealth of Nations" explored the social benefits of self-interest
  - Law-abiding self-interest is win-win: "It's not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard for their own interest"
- Smith attacked win-lose mercantilism: "We must always take heed that we buy no more from strangers than we sell them, for so should we impoverish ourselves and enrich them" (1549)

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- "When a country is losing many billions of dollars on trade with virtually every country it does business with, trade wars are good, and easy to win" — Trump (2018)

The First Welfare Theorem

Proposition (Arrow (1951) & Debreu (1951), 1940s folk result)

If $(\mathbf{p}, \mathbf{x})$ *is a competitive equilibrium of* $\mathcal{E}$, *and preferences are locally non-satiated, then* $\mathbf{x}$ *is socially efficient.*

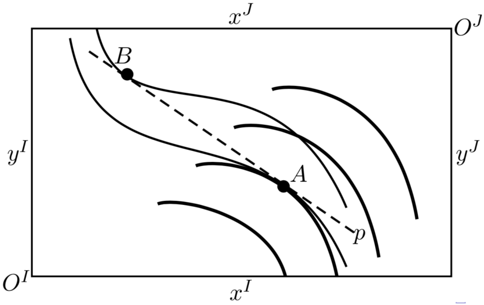
- Idea: If another allocation is better for all, strictly for Joe, then it costs everyone as much (at market price), and Joe strictly more. So it costs more than old allocation, and so more than the endowment.
- *Proof*: If $\mathbf{x}$ is socially inefficient, there is a feasible allocation $\mathbf{z}$ with $u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i)$ for all i, and $u^j(\mathbf{z}^j) > u^j(\mathbf{x}^j)$ for some j.
- Claim 1: $\mathbf{p} \cdot \mathbf{z}^i \geq \mathbf{p} \cdot \mathbf{x}^i$ for all i
    - Proof: If not, $\mathbf{p} \cdot \mathbf{z}^i < \mathbf{p} \cdot \mathbf{x}^i$ even though $u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i)$
    - By local nonsatiation, $\exists \mathbf{y}^i$ arbitrarily close to $\mathbf{x}^i$ (and so still affordable) but strictly preferred to $\mathbf{x^i}$, contrary to $\mathbf{x_i}$ optimal
- Claim 2: $\mathbf{p} \cdot \mathbf{z}^j > \mathbf{p} \cdot \mathbf{x}^j$
    - Proof: This follows since $\mathbf{x}^j$ is a utility maximizer for trader j
- Adding yields $\mathbf{p} \cdot \sum_{i=1}^n \mathbf{z}^i > \mathbf{p} \cdot \sum_{i=1}^n \mathbf{x}^i$.
- Since $\mathbf{p} \geq 0$, this contradicts $\sum_{i=1}^n \mathbf{z}^i \leq \sum_{i=1}^n \mathbf{x}^i$. $\square$

The Second Welfare Theorem

Proposition (Second Welfare Theorem)

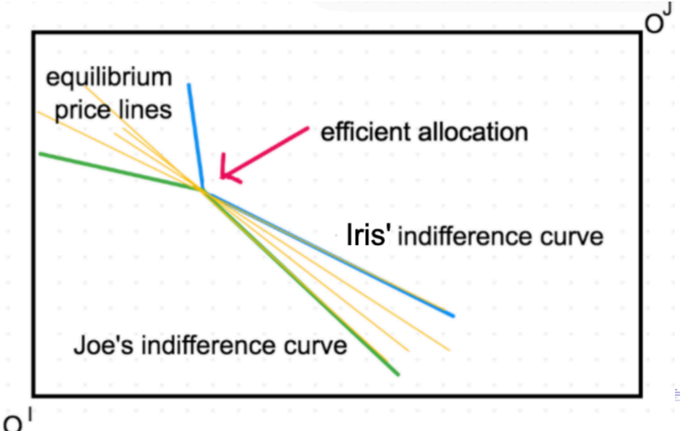
Assume that consumers have continuous, monotonic, and quasiconcave utility functions. If $\mathbf{x} \in \mathbb{R}_+^{Ln}$ is a socially efficient allocation, then there exists a price $\mathbf{p} \in \mathbb{R}_+^{L}$ and endowment $\bar{\mathbf{x}}$ such that $(\mathbf{x}, \mathbf{p})$ is competitive equilibrium of $\mathcal{E} = (\{u^i\}, \bar{\mathbf{x}})$.

- Proof logic uses duality for separation of convex sets.
- Important economics of a failure of convex preferences:



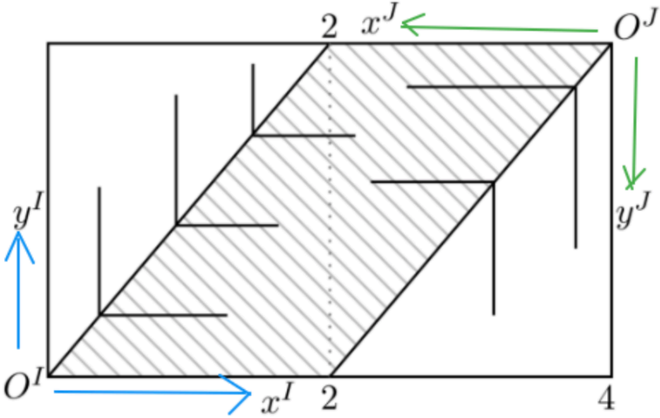
The Second Welfare Theorem: Price Multiplicity

- As in a double auction, equilibrium prices need not be unique.
- Nonuniqueness is less clear here, given an intensive margin
- Question: When are competitive prices unique?
  Answer: At least one consumer has smooth convex preferences
- Assume consumer indifference curves share a common "kink":



The Second Welfare Theorem: In-Class Thinker

- In our earlier Edgeworth Box, how do we decentralize any of the shaded efficient allocations?
  - Hint: What good is in excess supply?

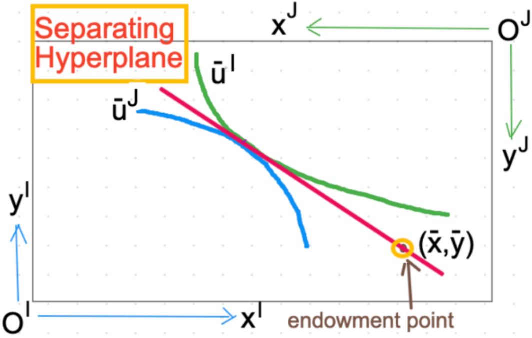


Review of Public Goods and Price/Quantity Constraints

- Govts like to constrain price & quantity, ignoring the margin.
  - With a token to transfer utility, and assuming a secondary market for the token, it creates deadweight loss triangles
  - With no token to transfer utility, some nontransferable currency emerges (queues, rent-seeking (grants!), violence).
    - This creates deadweight loss triangles and rectangles.
  - Model is "wronger" than usual — it's a reduced form game for queues and crime!

The Second Welfare Theorem: Topological Proof Idea

- The (Minkowski) Separating Hyperplane Theorem proof intuitively works for two traders
  - Minkowski taught Einstein in Zurich in late 1800s
  - 1908, he reformulated his 1905 special relativity as spacetime
  - 1909, sadly died at age 44 of appendicitis
- The Separating Hyperplane Theorem easily works for $n = 2$ — and is ugly mess for $n > 2$ consumers



- Pareto (1906/9) claimed a "proof".

MY Second Welfare Theorem Proof (2017)

- Let's parallel Shapley and Shubik's 1971 housing model proof
- So we not only prove the theorem, but interpret the prices
- Assume differentiable utility functions (my one simplification)
- *Proof:* At a socially efficient allocation $\mathbf{x}$, any Trader $\boxed{\text{Joe } j}$ maximizes his own utility, s.t. minimum others' utility from $\mathbf{x}$:

$$\max_{\mathbf{z}} u^j(\mathbf{z}^j) \quad \text{s.t.} \quad u^i(\mathbf{z}^i) \geq u^i(\mathbf{x}^i) \text{ for all } i \neq j$$
$$\textstyle\sum_i \mathbf{z}^i_\ell \leq \sum_i \mathbf{x}^i_\ell \text{ for } \ell = 1, \ldots, L \text{ (feasibility)}$$

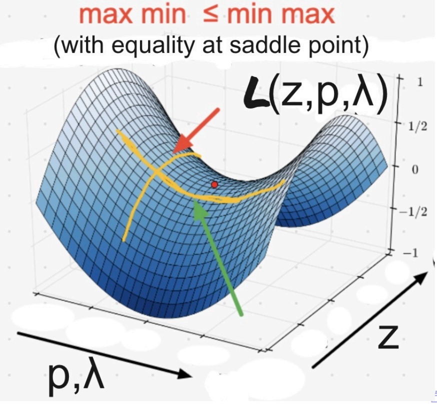
- As $\mathbf{x}$ is efficient, this maximum is realized at $\mathbf{z} = \mathbf{x}$.
  - The objective function, Joe's utility u^j, is quasiconcave
  - The constraint set is nonempty if no one is near a subsistence utility level (regularity condition on utility functions)
  - The constraint set is convex if $u^j(\mathbf{z}^i)$ is quasiconcave

$\Rightarrow$ Lagrangian has a saddle point for some multipliers $\lambda, \mathbf{p} \geq 0$

$$\mathcal{L}^j(\mathbf{z}, \mathbf{p}^j, \lambda^j) = u^j(\mathbf{z}^j) + \sum_{i \neq j} \lambda^j_i [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] + \sum_\ell p^j_\ell \left[\sum_i x^i_\ell - \sum_i z^i_\ell \right]$$

Second Welfare Theorem via Saddle Point Property

- Second Welfare Theorem says ∃ prices giving an equilibrium
- Unlike the 1951 proofs by Arrow and Debreu, this offers a computer recipe for finding prices in an economy!
- Recall: A saddle point is a max for **z** and a min for multipliers



Second Welfare Theorem Proof via Saddle Point Property

- By the saddle point property, $(\mathbf{x}, \mathbf{p}^j, \lambda^j)$ is a maximum of $\mathcal{L}^j(z, \mathbf{p}^j, \lambda^j)$ in $\mathbf{z}$, and a minimum of $\mathcal{L}^j(\mathbf{x}, \mathbf{p}, \lambda)$ in $(\mathbf{p}, \lambda)$.
- By the maximum property, $\mathcal{L}^j(\mathbf{z}, \mathbf{p}^j, \lambda^j) \leq \mathcal{L}^j(\mathbf{x}, \mathbf{p}^j, \lambda^j)$ for all z:

$$\Rightarrow \quad u^j(\mathbf{z}^j) + \textstyle\sum_{i \neq j} \lambda_i^j [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] + \mathbf{p}^j \cdot [\mathbf{x}^j - \mathbf{z}^j] \leq u^j(\mathbf{x}^j). \quad (\bigstar)$$

- Claim: $u^j(\mathbf{z}^j) > u^j(\mathbf{x}^j) \Rightarrow \mathbf{p}^j \cdot \mathbf{z}^j > \mathbf{p}^j \cdot \mathbf{x}^j$ for all j
  - If so, then no trader j can afford a bundle $\mathbf{z}^j$ with more utility than $\mathbf{x}^j$ at price $\mathbf{p}^j \Rightarrow x^j$ is an optimal bundle for trader j at p^j
  - Proof of Claim: Since $u^j(\mathbf{z}^j) > u^j(\mathbf{x}^j)$, then $(\bigstar)$ implies

$$\textstyle\sum_{i \neq j} \lambda_i^j [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] + \mathbf{p}^j \cdot [\mathbf{x}^j - \mathbf{z}^j] < 0$$

$$\Rightarrow \mathbf{p}^j \cdot [\mathbf{z}^j - \mathbf{x}^j] > \textstyle\sum_{i \neq j} \lambda_i^j [u^i(\mathbf{z}^i) - u^i(\mathbf{x}^i)] \geq 0$$

- $(\mathbf{x}, \mathbf{p})$ is competitive equilibrium if one price p works for all j.
- We next prove that $\mathbf{p}_\ell^j = c_j \mathbf{p}_\ell$, for $\ell = 1, \ldots, L$ & some $c_j > 0$

  $\Rightarrow$ A common price for all j allows the revealed preference proof
  - Trader j's utility is obviously scalable $\Rightarrow$ so too is p^j

Offline: Common Price in Second Welfare Theorem Proof

- Optimality in z^i_ℓ and z^j_ℓ for all traders $i \neq j$ yield the FOC's:

$$\begin{aligned}
\frac{\partial}{\partial z^j_\ell}\mathcal{L}^j(\mathbf{z},\mathbf{p}^j,\lambda^j) &= \frac{\partial}{\partial z^j_\ell}u^j(z^j) - p^j_\ell = 0 \\
\frac{\partial}{\partial z^i_\ell}\mathcal{L}^j(\mathbf{z},\mathbf{p}^j,\lambda^j) &= \lambda^j_i \frac{\partial}{\partial z^i_\ell}u^i(z^i) - p^j_\ell = 0
\end{aligned}$$

- Equating p^j_ℓ for traders $i \neq j$, we get: $\lambda^j_i = \frac{\partial}{\partial z^j_\ell}u^j(z^j) \Big/ \frac{\partial}{\partial z^i_\ell}u^i(z^i)$
- Equate Planner's MRS between any traders i, j across goods ℓ

$$\frac{\partial}{\partial z^i_{\ell_1}}u^i(z^i) \Big/ \frac{\partial}{\partial z^j_{\ell_1}}u^j(z^j) = \frac{\partial}{\partial z^i_{\ell_2}}u^i(z^i) \Big/ \frac{\partial}{\partial z^j_{\ell_2}}u^j(z^j)$$

$\Rightarrow$ Starting with agent k rather than j, the price ratio is the same:

$$p^k_{\ell_1}/p^k_{\ell_2} = p^j_{\ell_1}/p^j_{\ell_2}$$

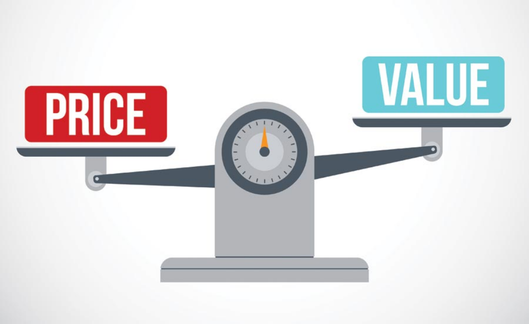
$\Rightarrow$ Multipliers are $\mathbf{p}^j_\ell = c_j \mathbf{p}_\ell$, some $c_j > 0$, in Lagrangian for all j

Prices as Shadow Values

- The price p_ℓ is the multiplier on the constraint

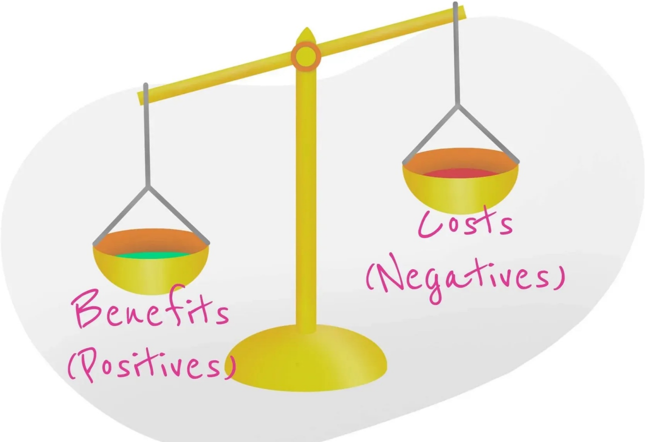
$$\sum_i x_\ell^i - \sum_i z_\ell^i \geq 0$$

- A good's price in a competitive equilibrium is its social shadow value
- If price is not (marginal) value, the equilibrium is inefficient
- Price is indeterminant up to a constant, as marginal utility is!



Fun Aside: Economics is Cost Benefit Analysis Run Amuck

- We assume people optimize: they do actions with $B > C$
  - With an intensive margin, they do the action with max $B - C$
- Welfare Theorems (for competitive markets & private goods)
  - private cost benefit analysis $\Leftrightarrow$ societal cost benefit analysis
- Coase Theorem: clear property rights $\Rightarrow$ welfare theorem



Excess Demand Functions

- Strictly convex preferences $\Rightarrow$ unique demands $x^i_\ell(p)$
- Trader i's excess demand (net demand): $ED^i_\ell(\mathbf{p}) = x^i_\ell(p) - \bar{x}^i_\ell$
- The market excess demand for x_ℓ is $ED_\ell(\mathbf{p}) = \sum_{i=1}^n ED^i_\ell(\mathbf{p})$
- Markets clear in a competitive eq $(\mathbf{x}(\mathbf{p}), \mathbf{p})$: $ED_\ell(\mathbf{p}) = 0 \ \forall \ell$

Lemma (Walras Law)

If traders consume their entire income at allocation $\mathbf{x}(p)$, then the market value of net excess demand vanishes: $\sum_{\ell=1}^L p_\ell ED_\ell(\mathbf{p}) = 0$.

- *Proof*: Trader i's budget constraint $\mathbf{p} \cdot \mathbf{x^i}(\mathbf{p}) \equiv \mathbf{p} \cdot \bar{\mathbf{x}}^i$:

$$\sum_{\ell=1}^L p_\ell ED^i_\ell \equiv \sum_{\ell=1}^L p_\ell [x^i_\ell(p) - \bar{x}^i_\ell] \equiv 0 \qquad \square$$

- Only $L-1$ independent equations $ED_\ell(p) = 0$ (last clears by Walras)
- Demand is homogeneous of degree zero in (income, prices) $\Rightarrow$ prices have one degree of freedom: $ED(p) \equiv ED(tp)$ for all $t > 0$
  - Normalize one price to one — i.e. good is numeraire (currency)
  - Or, we can ask that all prices sum to one (we do this later)

$\Rightarrow$ Equilibrium is $L-1$ nonlinear equations in $L-1$ prices

Existence Using Excess Demand Functions: $L = 2$ Goods

$\Rightarrow$ Measure the *price ratio* $p = p_y/p_x$ of y in units of numeraire x.

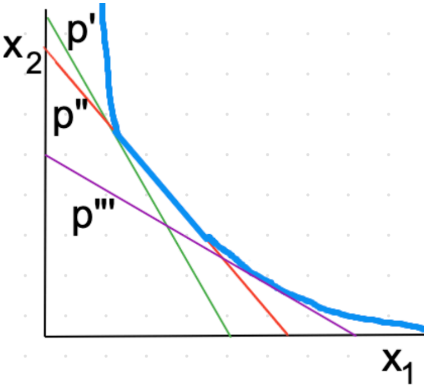
$\Rightarrow$ Equilibrium is 1 equation in 1 unknown: $ED_x(p) = 0$.

Theorem (Baby General Equilibrium Existence)

Assume two goods x and y. Trader i has monotone and strictly convex preferences, and owns a nonzero endowment $(\bar{x}^i, \bar{y}^i)$. There exists a stable Walrasian competitive equilibrium $(\mathbf{x}, \mathbf{y}, p)$.

- *Proof* **sketch:** With strictly convex preferences, each trader i has a unique optimal consumption bundle $x^i(p)$ at any $p > 0$.
- Optimizers upper hemicontinuous in p (Theorem of the Max)
  - $\Rightarrow$ Unique optimizer $x^i(p)$ is continuous in p
  - $\Rightarrow$ $ED_x(p)$ is a continuous function
    - Monotone preferences $\Rightarrow ED_x(0) < 0 < ED_x(\infty)$
    - Intermediate Value Theorem $\Rightarrow ED_x(p) = 0$, for some $p > 0$.
- At least one zero of $ED_x(p) = 0$ is stable, crossing - to +
- General proof for $L \geq 2$ goods awaits model with production

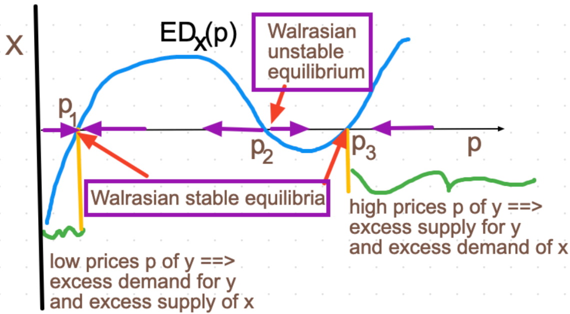
Weakly Convex Preferences $\not\Rightarrow$ Unique Demands



- An interval of demands all solve the optimum at some prices
- There may be no demand function!
- We also revisit this issue later in our general theory

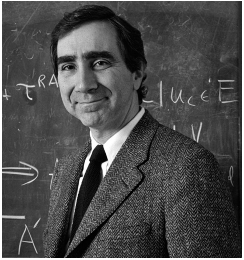
Existence and Stability of Competitive Equilibrium

- Monotone preferences $\Rightarrow ED_x(0) < 0 < ED_x(\infty)$



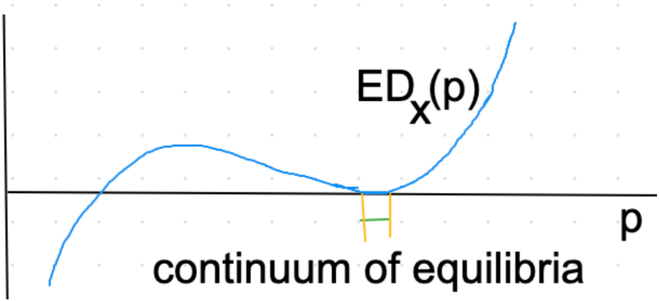
- Wilson's Oddness Theorem (1971)
  - A game with finitely many players and actions has an odd number of Nash equilibria, for "generic" payoffs
- I soon show that with perfect competition, a market is a game where the Walrasian auctioneer is a player picking the price
- *For generic payoffs, a market has an odd number of equilibria*
  - Proof: Like existence, this is visually clearly with $n = 2$ goods

Hugo Sonnenschein (1940–2021) [My Academic Grandpa]



- Hugo was adviser to the 1980s Game Theory Renaissance
- Macro *hugely* relies on representative agent models. Smart?
- The Sonnenschein-Debreu-Mantel Theorem (1972/1974) says that GE has few implications with heterogeneous agents
  - *Any continuous function that obeys homogeneity and Walras Law is an excess demand of some economy — for enough consumers with some utility functions and endowments.*
    - Debreu showed you only needed as many consumers as goods
    - Mantel: We can assume homothetic $\succ$ (eg Cobb-Douglas)
    - This may be the prettiest result in general equilibrium theory!

Local Uniqueness of Equilibria



- *Comparative statics are meaningless if we do not know which equilibrium we refer to. So multiple equilibria are problematic.*
- Worse yet: Could there be an interval of equilibria?
- Debreu proved an excess demand curve "rarely" vanishes on an price interval — only for a "null" set of endowments
  - Null is more rare than probability zero (eg rationals are zero measure, but not null, since they're dense in the real line)
  - Proof via Sard's Theorem (1942 differential topology result)

Example: Cobb Douglas Preferences & $L = 2$ Goods

- Utilities: Iris $u^I(x,y) = x^\alpha y^{1-\alpha}$ and Joe $u^J(x,y) = x^\beta y^{1-\beta}$
- Endowments: $(\bar{x}^I, \bar{y}^I)$ and $(\bar{x}^J, \bar{y}^J)$.
- Incomes: $w^I(p) = \bar{x}^I + p\bar{y}^I$ and $w^J(p) = \bar{x}^J + p\bar{y}^J$
  - The wealth (i.e. endowment value) varies as the price p moves
- Cobb-Douglas demands: $x^I(p,w) = \alpha w^I$ & $x^J(p,w) = \beta w^J$
- Market excess demand:

$$ED_x(p) = \left(\alpha w^I(p) - \bar{x}^I\right) + \left(\beta w^J(p) - \bar{x}^J\right)$$

- Walras $\Rightarrow$ It suffices to clear the x market:

$$ED_x(p^*) = 0 \quad \Rightarrow \quad p^* = \frac{\bar{x}^I(1-\alpha) + \bar{x}^J(1-\beta)}{\alpha\bar{y}^I + \beta\bar{y}^J}$$

- The competitive price p^* reflects preferences and endowments
  - falls in α, β (greater love of x by either trader raises its price)
  - rises if $\bar{x}^I$ or $\bar{x}^J$ rises (gold discoveries led to inflation)

High value $\Rightarrow$ scarcity and convex preferences near 0

Fun: Gold and the Wizard of Oz (1900 book, 1939 movie)

- 1896: William Jennings Bryan (Democratic nominee for president) condemned gold standard in "Cross of Gold" speech

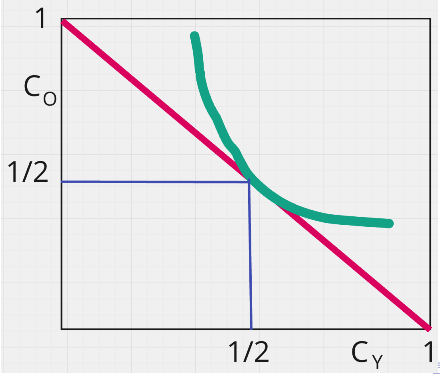
- Dorothy follows the Yellow-brick road (gold standard) to reach the Wizard (President McKinley).
  - Her silver slippers (ruby in the movie) help her get home

Samuelson's 1958 OLG model

- The title is weird!
- "Let us assume that men enter the labor market at about the age of twenty. They work for forty-five years or so and then live for fifteen years in retirement. (As children they are part of their parents' consumptions, and we take no note of them.)"
- "Break each life up into thirds: men produce one unit of product in period 1 and one unit in period 2; in period 3 they retire and produce nothing. (No one dies in midstream.)"
- "each man's tastes can be summarized by an ordinal utility function of the consumptions of the three periods of his life: $U = U(C_1, C_2, C_3)$."
- "let $R_t = 1/(1 + i_t)$ be the discount rate between goods"
- He solves an example with $U(C_1, C_2, C_3) = \log(C_1) + \cdots + \log(C_3)$
  - Stationary solution: price R of C_t in terms of C_{t+1} is $R = (3 + \sqrt{13})/2$

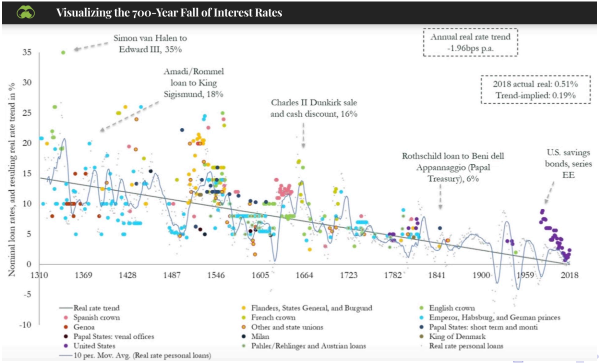
Our Dumbed-Down OLG model in an Edgeworth Box

- Assume first young then old (endowed with 1) and retired, equal mass
- max $\sqrt{C_Y C_O}$ s.t. $C_Y + RC_O = 1$
- Cobb-Douglas demands: $C_Y = 1/2$ and $C_O = 1/(2R)$
- Markets clear: $C_Y + C_O = 1$
- Solution: $R + 1 = 2R$ implies $R = 1$



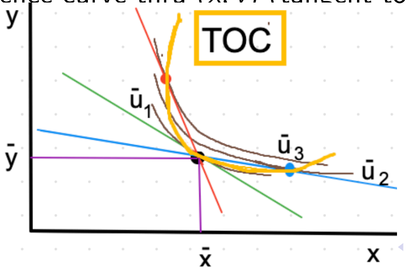
Edgeworth Box as an Intergenerational Model (Prelim, '21)

- Iris and Joe can capture
  - representative traders in 2 countries (International trade theory!)
  - the same person in consecutive periods (savings model)
    - $1 + r$ is the price ratio of consumption today to tomorrow
  - adjacent generations in an intergenerational model (constant growth)
    - 2021 Prelim: Why have interest rates fallen so much?
    - Idea: *longevity reduces interest rates* (care about future more)
    - Population growth (births or immigration) lowers interest rates



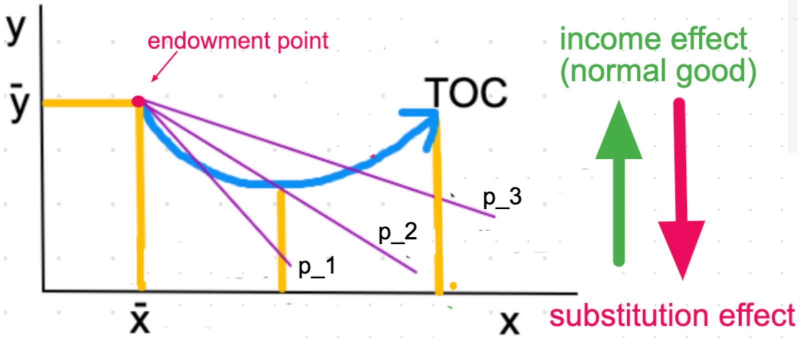
Trade Offer Curves: Consumer Theory to Trade Theory

- The trade offer curve (TOC) plots optimal consumption allocations as prices vary, for fixed endowments.
- TOC is the locus of indifference curve tangencies to the price line through the endowments $(\bar{x}, \bar{y})$:
  $=$ price-consumption curve (consumer theory from initial bundle)
    - Note: Trade theory overlaps heavily with consumer theory
  - *TOCs are like best reply graphs in game theory*
- Claim: With $L = 2$ goods, TOC is tangent to the indifference curve through the endowment, and "more curved" than it
- Proof: $\exists$ indifference curve thru $(\bar{x}, \bar{y})$ (tangent to some price line)



Offline: Income Elasticities and the TOC

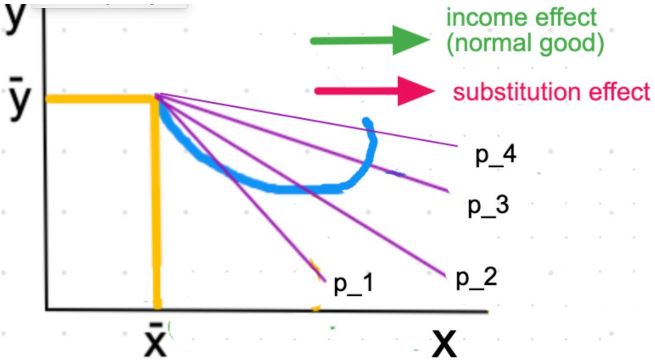
- The TOC and three price lines $p_1 < p_2 < p_3$ are depicted



- The TOC can be nonmonotone, despite monotone preferences
  - As the price p of y in terms of x rises, substitution effect: $y \downarrow$.
  - Along the TOC, Iris is a net supplier of y

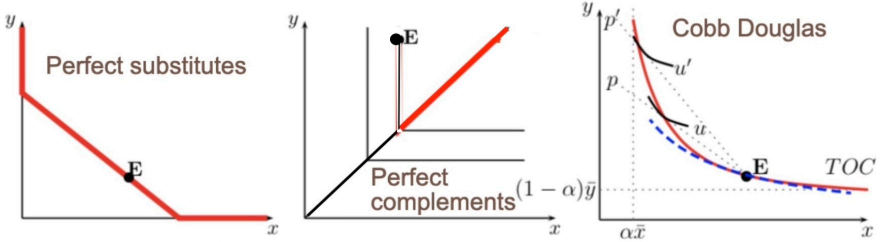
  $\Rightarrow$ As the price of y rises, real income (endowment value) rises
    - If y is a normal good, then the TOC can fall or rise
    - If y is an inferior good, then the TOC is strictly falling

Offline: Backward Bending TOC Requires Inferiority



- Price of x in terms of y falls as $p\uparrow \Rightarrow$ substitution effect: $x\uparrow$
- As the price of y rises, Iris' real income rises
    - If x is a normal good, then the TOC moves right
    - If x is an inferior good, the TOC can turn back (but need not)

Classic Trade Offer Curve for Typical Preferences



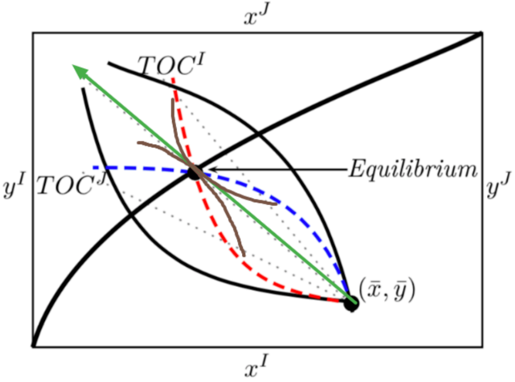
- Perfect substitutes, perfect complements, and Cobb Douglas
- Assume Cobb-Douglas utility $u(x, y) = x^{\alpha} y^{1-\alpha}$
- Recall that the TOC is the locus of indifference curve tangencies to the price line through the endowments $(\bar{x}, \bar{y})$:

$$\frac{(1-\alpha)x}{\alpha y} = MRS = p = \frac{\bar{x}-x}{y-\bar{y}} \Rightarrow y(x) = \frac{(1-\alpha)\bar{y}x}{x-\alpha\bar{x}}$$

- The TOC starts at $y(\bar{x}) = \bar{y}$, for there is always a price for which it is efficient to consume the endowment.

Competitive Equilibrium via Trade Offer Curves

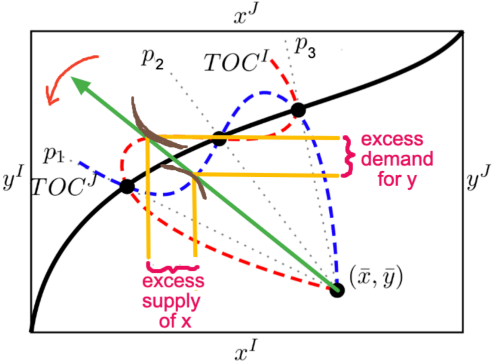
- A crossing of TOC^I and TOC^J is a competitive equilibrium, since each trader optimally chooses that bundle
  - Analogously, for a normal form game, the intersection of best reply functions is a Nash equilibrium
  - This finds the competitive in goods space, whereas excess demand approach finds it in price space



- ∃ *unique equilibrium (TOC crossing) if all goods are normal*

Non-Uniqueness: Trade Offer Curves

- Assume some good is so inferior that TOC's multiply cross
- The absolute slope of the price line is $p_x/p_y = 1/p$
- There are three equilibrium prices (of y): $p_1 > p_2 > p_3$
- Claim: p_1 and p_3 are Walrasian stable, and p_2 is not:
  - If $p \in (p_2, p_1)$, then excess demand for $y \Rightarrow p \uparrow p_1$
- With multiple equilibria, alternating equilibria are stable



- TOC Aside: normal goods $\Rightarrow$ unique competitive equilibrium

Gross Substitutes Implies Uniqueness

- Demand has the gross substitutes (GS) property if an increase in price p_k raises the demand of all other goods x_ℓ, for $\ell \neq k$.
  - Falling best reply in a submodular game ("strategic substitutes"): other actions $\uparrow \Rightarrow$ best reply$\downarrow$. Unique equilibrium? It's complicated
  - ★ Rising best reply in a supermodular game ("strategic complements"): *Multiple equilibria can arise.*

Proposition (Uniqueness)

If the aggregate excess demand function satisfies gross substitutes, the economy has at most one competitive equilibrium

- Proof: Let $ED(p) = ED(p') = 0$ for p, p' not linearly dependent.
- Scale price vectors so that $p_\ell \geq p'_\ell$ for all ℓ, and $p_k = p'_k$ for some k
  - $p = (48, 12, 4)$ and $p' = (8, 4, 2) \Rightarrow$ scale p' to $\hat{p} = (16, 8, 4)$
  - Obviously, demand is the same at $\hat{p}$ and p' (by homogeneity)
- Change from $\hat{p}$ to p in $L - 1$ steps, raising $\hat{p}_\ell$ for each $\ell \neq k$.
  - Raise $\hat{p}_2$ from 8 to $12 = p_2$, and then $\hat{p}_1$ from 16 to $48 = p_1$
- Since aggregate demand x_k rises each step, $ED_k(p) > ED_k(p') = 0$.

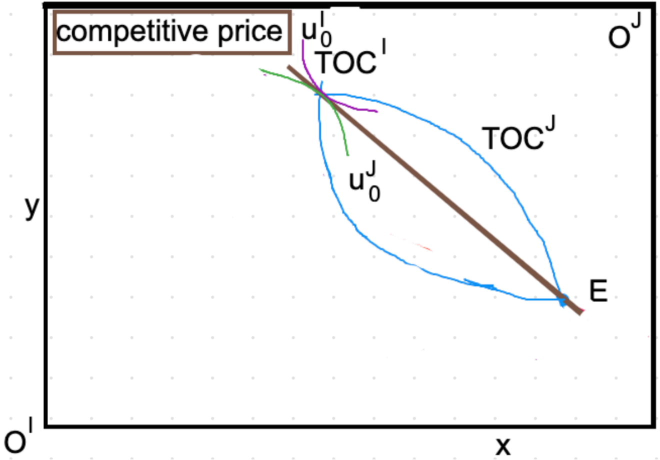
Begin with the end in mind

Review of Multimarket Equilibrium

- Two subtle features of economics:
  1. how price & quantity adjust in to clear markets (partial eq'm)
  2. how one market impacts other markets (general eq'm)
- Why does an equilibrium exist? Might prices forever adjust?
- A baby fixed point theorem "proves" it in exchange economies
- Excess demand functions $\Rightarrow$ a stable equilibrium exists
- Welfare theorem: competitive markets are socially efficient
- Sonnenschein Theorem: excess demand functions are quite arbitrary
- Uniqueness $\Leftarrow$ Gross substitutes condition
- Trade offer curves link uniqueness to normality of goods

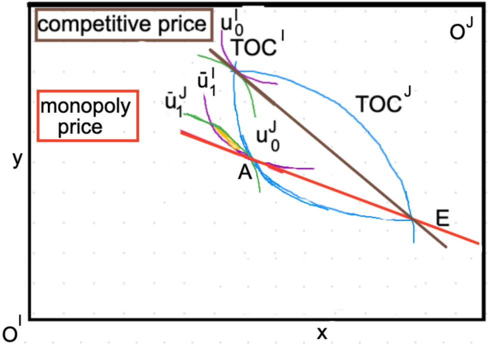
Competitive Equilibrium in the Edgeworth Box

- Start with a competitive equilibrium with two goods, in which Joe sells y to the market and buys x



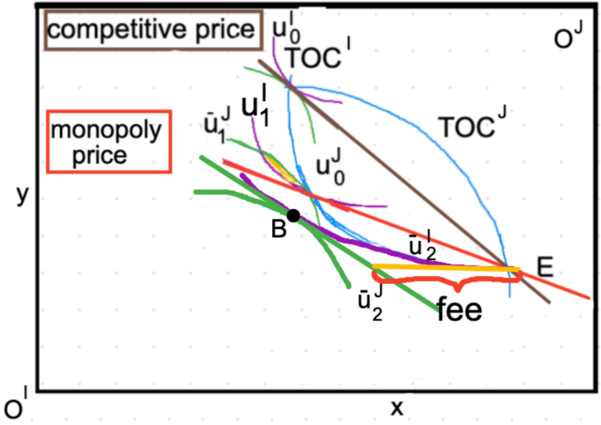
Monopoly Joe Replaces the Walrasian Auctioneer

- Joe seeks his *highest indifference curve on Iris's TOC*: $\bar{u}_1^J > \bar{u}_0^J$
  $\Rightarrow$ The indifference curve $\bar{u}_1^J$ is tangent to TOC^I at A
- He sets a higher price ratio or y to x, since he sells y
- Finally, we can see the monopoly inefficiency:
  - Proof: The (red) price line slices thru TOC^I, and so thru $\bar{u}_1^J$
  - But indifference curve $\bar{u}^I$ is tangent to the (red) price line at A
  - $\exists$ gains from trade (slender orange lens)



Monopoly Kingpin Joe Sets a Two Part Tariff

- Assume Joe can sets a two part tariff, i.e. a fixed trading fee, and a linear price of y to x (like Disney prices)
- Joe now secures an even higher utility $\bar{u}_2^J > \bar{u}_1^J$
- Omnipotent monopoly is efficient: B is on the contract curve!

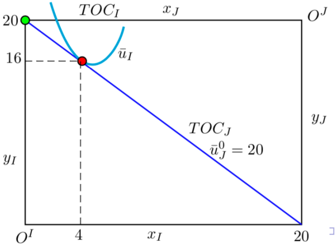


Offline: Monopoly in the Edgeworth Box Practice Exercise

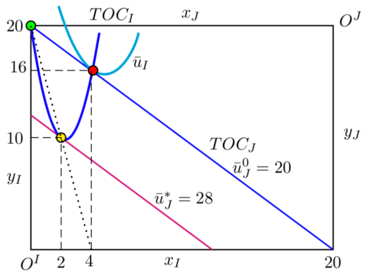
- EXAMPLE (TONO CARRASCO):
- $u^J(x, y) = x + y$ and $\bar{x}^J = 20$ and $\bar{y}^J = 0$.
- $u^I(x, y) = x(9 - x) + y$ and $\bar{x}^I = 0$ and $\bar{y}^I = 20$.
- Find competitive equilibrium, and best linear pricing monopoly, and best two part pricing monopoly
- Tono's solution and graphics are posted in canvas

Offline: Monopoly in the Edgeworth Box Example Solved

- Competitive Equilibrium
  - Iris' utility $u^I(x,y) = x(9-x) + y$ with $\bar{x}_I = 0$ and $\bar{y}_I = 20$.
    - Her indifference curves have the form: $y = \bar{u} + x(x-9)$
  - Joe's utility $u^J(x,y) = x + y$ with $\bar{x}_J = 20$ and $\bar{y}_J = 0$.
  - Joe has constant $MRS = 1 \Rightarrow$ equilibrium price of y is $p = 1$
    - advantage of one consumer having perfect substitutes utility
  - Iris has quasi-linear preferences and is linear in y
    $\Rightarrow$ pick y as numeraire, with relative price $\pi = 1/p$ of x
  - Iris equates $9 - 2x_I \equiv u^I_x / u^I_y = \pi = 1 \Rightarrow x_I = 4$ and $y_I = 16$
  - So Joe demands the residual $x_J = 16$ and $y_J = 4$.
  - First Welfare Theorem: The outcome is efficient (tangency)



Offline: Monopoly in the Edgeworth Box Example Solved



- Joe is a linear pricing monopolist $\Rightarrow$ sets the price π of x
- Iris is a price taker $\Rightarrow$ demands $x_I(\pi) = (9 - \pi)/2$ ($\bigstar$)
  $\Rightarrow$ $y_I(\pi) = 20 - \pi(9 - \pi)/2$, from her budget constraint ($\bar{y}_I = 20$)
  - *We now solve for the (quadratic) trade offer curve of Iris.*

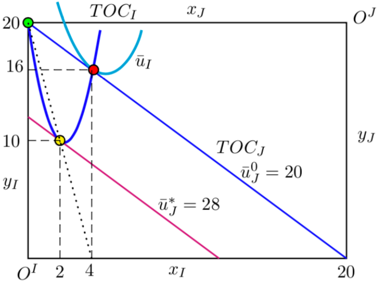
  $\Rightarrow$ TOC$_I$: $y = 20 - \pi x = 20 - (9 - 2x)x$ by budget constraint, ($\bigstar$)
- Joe maximizes indirect utility

$$V_J(\pi) = x_J + y_J = [20 - x_I(\pi)] + [20 - y_I(\pi)] = 20 - (9 - \pi)/2 + \frac{1}{2}\pi(9 - \pi)$$

$\Rightarrow$ FOC: $1 + 9 - 2\pi = 0 \Rightarrow \pi = 5$.

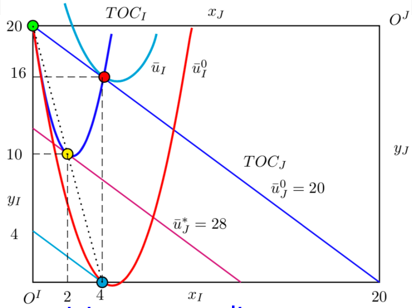
- Joe sets higher than competitive price π for his endowed good

Offline: Monopoly in the Edgeworth Box Example Solved



- Iris' demands
  - $x_I(\pi) = (9 - \pi)/2 = (9 - 5)/2 = 2$
  - $y_I(\pi) = 20 - \pi(9 - \pi)/2 = 5(9 - 5)/2 = 10$
- Joe's demands are the residual: $x_J = 18$ and $y_J = 10$.
- Joe's utility rises from $4 + 16 = 20$ to $18 + 10 = 28$
- Iris's utility falls from $4(9-4) + 16 = 36$ to $2(9-2) + 10 = 24$.
  - This still beats Iris's endowment utility of 20.
- Inefficiency of monopoly: Joe's *MRS* is constant at one, whereas Iris ends up with $MRS = u_x^I/u_y^I = 9 - 2x_J = 9 - 4 = 5$.

Offline: Monopoly in the Edgeworth Box Example Solved



- Joe is a non-linear pricing monopolist
- Iris' reservation utility is $u_0^I(0,20) = 20$ at endowment $(0,20)$.

$\Rightarrow$ Iris needs $u^I(x,y) = x(9-x) + y \geq 20$

- Joe maximizes welfare given Iris's demands $(x, 20 - x(9-x))$

$$[20 - x] + [20 - y] = (20 - x) + x(9 - x) \quad \Rightarrow -1 + 9 - 2\hat{x} = 0$$

- So Iris consumes $(\hat{x}_I, \hat{y}_I) = (4, 0)$ and Joe $(\hat{x}_J, \hat{y}_J) = (16, 20)$
- Two part tariff: Iris pays a fee $y = 16 - \varepsilon$, then a price $\pi = 1$
- Inefficiency of monopoly: vanishes

Beyond Markets: Cooperative Games and Core Theory

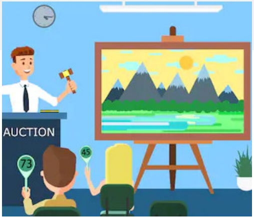
- We briefly return to a world with just an extensive margin
  - This framework subsumes TU matching as a special case
  - It includes markets with market power, and public goods
  - We derive a coalitional rationale for competitive equilibrium!
- Many coalitions of people can form for greater good
  - Examples: political parties, military alliances, criminal gangs
  - Examples: university friends? People you are web-linked to?
    - This sheds light on network economics (popular at Stanford)
  - We ignore optimization, but allow many extensive margins!
  - We focus entirely on coalitional participation constraints
- Computer scientist Donald Gillies (1928–75) created it in his 1953 PhD thesis (Gillies was at Princeton with John Nash)

Core Theory

- $N =$ set of all players (we call its size N too)
- A coalition is a group of players $S \subseteq N$ (grand coalition)
- Players earn payoff vector $u \in \mathbb{R}^N$ — called the *imputation*
- Coalition $S \subseteq N$ has value $v(S)$, where $v(\varnothing) = 0$
  - i.e. coalition S can secure payoff $v(S)$ by itself, ignoring $i \notin S$
  - v may require an optimization by players $i \in S$
    - This is usually unmodelled.
    - But in canvas public goods application, we derive values!
  - If a coalition $S \subset N$ cannot form, simply set $v(S) = -\infty$
  - Pairwise matching model: $v(S) =$ match payoff if $|S| = 2$

  $\Rightarrow$ The TU matching model is also a coalitional game
- Coalition S blocks payoff imputation $u \in \mathbb{R}^N$ if $\sum_{i \in S} u_i < v(S)$
  - Core constraints $\sum_{i \in S} u_i \geq v(S)$ reflect competitive forces
- The core is all unblocked feasible payoffs u: $\sum_{i \in N} u^i = v(N)$
  - We need to support the grand coalition payoff:
  - All coalitions are threat points only, via the core constraints!

Core Theory and Auctions



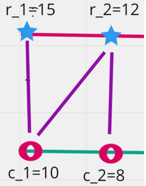
- Seller S values painting at 100
- buyers B_1, B_2 value it at $120, 150$
  - $\Rightarrow$ $v(B_1) = v(B_2) = v(B_1, B_2) = 0$
  - $\Rightarrow$ $v(S) = 100$, $v(B_1, S) = 120$, $v(B_2, S) = v(B_1, B_2, S) = 150$
- SOLUTION:
  - CS payoffs of B_1, B_2 are b_1, b_2 & PS + cost payoff of S is p
  - IR core constraints: $b_1 \geq 0, b_2 \geq 0, p \geq 100$.
  - Pairwise core constraints (competitive forces):

$$p + b_1 \geq 120, p + b_2 \geq 150, b_1 + b_2 \geq 0$$

  - Grand coalition earns $v(B_1, B_2, S) = p + b_1 + b_2 = 150$.
  - Core (competition): $b_1 = 0$ & $120 \leq p \leq 150$ & $b_2 = 150 - p$.
- Auction finds the max price the seller can guarantee herself
  - I.e., what's the best take it or leave it offer of buyers to seller?

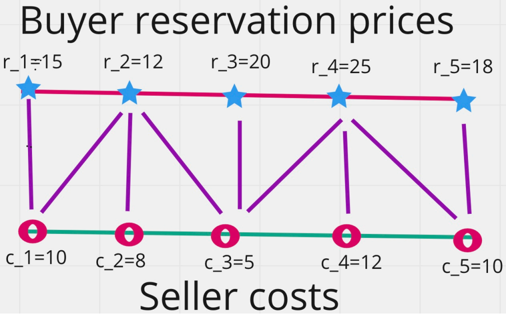
$\Rightarrow$ Highest value bidder wins; expects to pay 2^{nd} highest value

Core Theory and Markets with Missing Trade Links

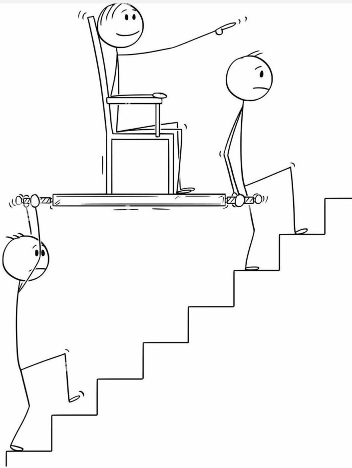


- Add a second seller
- Two buyers, two sellers: Buyer 1 and Seller 2 cannot trade
- Coalition values:
  - $v(c_1) = 10, v(c_2) = 8, v(r_1, c_1) = 15, v(r_2, c_1) = 12 = v(r_2, c_2)$
- Payoffs:
  - $p_1 \geq 10, p_2 \geq 8, b_1 + p_1 \geq 15, b_2 + p_1 \geq 12, b_2 + p_2 \geq 12$
- Missing link invalidates the law of one price $(10 \leq p \leq 12)$
  - Example: $p_1 = 13$ and $p_2 = 11$ are competitive prices
- Law of One Price: If all buyers and sellers are connected, deduce $p_1 = p_2 = p$ from the core constraints.
  - Proof Hint: Just use pairwise core constraints, and grand coalition value equality $p_1 + p_2 + b_1 + b_2 = 27$
- Research Q: What does the core say about middlemen profits?
- Research Q: What are the gains to forming more links?

Offline: Find Core Prices



The Empty Core Possibility



- A table must be carried by ≥ 2 students
- The value of this job is 50.
- There are three possible table carriers
- The grand coalition yields payoff
  $u_1 + u_2 + u_3 = 50 = v(1, 2, 3)$
  (i.e. the value of moving the table)
- IR core constraints: $u_i \geq v(i) = 0$.
- Pairwise core constraints:

$$
\begin{aligned}
u_1 + u_2 &\geq v(1,2) = 50 \\
u_2 + u_3 &\geq v(2,3) = 50 \\
u_1 + u_3 &\geq v(1,3) = 50
\end{aligned}
$$

- Summing: $u_1 + u_2 + u_3 \geq 75 > 50 = v(1, 2, 3) \Rightarrow$ *empty core!*

Illustrating the Core in a Modified Table Example

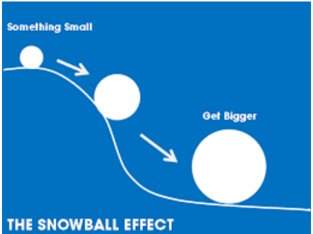
- Q: Why is the core empty?
- A: Increasing the coalition size here produces values $0, 50, 50$
  - Marginal increments must increase.
  - Here, the third student is always useless.
  - $\Rightarrow$ He competes away payoffs from the coalition of two students
- Voting game parallel: Two of three voters decide a policy.
  - This yields a value function like the table example
- Each core constraint reflects a competitive force: So too much competition is bad if the values of coalitions do not grow
  - Sub-coalitions can excessively undermine the grand coalition
- Lester Telser: Railroads went bankrupt due to competition after spending great fixed costs
- Cartels and Unions: deviations can undermine their power
  - "Right to work" laws allow firms to form coalitions with subgroups of employees

Properties of Transferable Utility Cooperative Games

- *Monotone:* $S \subseteq T \Rightarrow v(S) \leq v(T)$
- *Supermodular:* $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\ \forall S, T$
- *Superadditive*: $v(S \cup T) \geq v(S) + v(T)$ when $S \cap T = \emptyset$.
  - A supermodular valuation implies increasing returns to size.
  - Supermodular $\Rightarrow$ superadditive, if $v(0) = 0$
  - A convex game has a supermodular game value
- Shapley's Claim: v is supermodular if and only if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \quad \forall S \subseteq T \subseteq N \backslash \{i\}, \forall i \in N$$

- "Snowballing effect" emerges: *incentives for joining a coalition increase in its size* $\Rightarrow$ precludes table carrying example!



Convex Games: Instructive Proof (Peruse Offline)

Theorem (Bondareva-Shapley)

A convex game has a non-empty core.

- Key idea: the core is not empty iff $v(N)$ is at least

$$\min \textstyle\sum_i u_i \qquad \text{subject to } (\bigstar): \quad \textstyle\sum_{i \in S} u_i \geq v(S) \quad \forall S \neq N$$

- Shapley-Shubik solved a similar minimization for matching
- We show that convexity ensures this inequality for $v(N)$.
- $(\bigstar)$ barely holds if Mr. i is paid his marginal addition $u_i = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$ to $S = \{1, \ldots, i-1\}$
- Claim: The payoff $u = (u_1, \ldots, u_N)$ is in the core, i.e. no coalition $A_k = \{i_1, \ldots, i_{k-1}\}$ blocks it, where $i_1 < \cdots < i_k$

$$\begin{aligned}
\textstyle\sum_{j=1}^k u_{i_j} &= \textstyle\sum_{j=1}^k [v(\{1, \ldots, i_j\}) - v(\{1, \ldots, i_j - 1\})] \\
&\boxed{\geq} \textstyle\sum_{j=1}^k [v(\{i_1, \ldots, i_j\}) - v(\{i_1, \ldots, i_{j-1}\})] \\
&= v(\{i_1, \ldots, i_k\})
\end{aligned}$$

Convex Games: Instructive Proof (Peruse Offline)

Theorem (Bondareva-Shapley)

A convex game has a non-empty core.

- Proof of inequality $\boxed{\geq}$ for the payoff vector $u = (u_1, \ldots, u_N)$ with $u_i = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$
- Let coalition $A_k = \{i_1, \ldots, i_{k-1}\}$. By supermodular inequality

$$\begin{aligned}
\textstyle\sum_{j=1}^k u_{i_j} &= \textstyle\sum_{j=1}^k [v(\underbrace{\{1, \ldots, i_j\}}_{B_j \cup i_j = S \cup T}) - v(\underbrace{\{1, \ldots, i_j - 1\}}_{B_j = S})] \\
&\geq \textstyle\sum_{j=1}^k [v(\underbrace{\{i_1, \ldots, i_j\}}_{A_j = T}) - v(\underbrace{\{i_1, \ldots, i_{j-1}\}}_{A_{j-1} = S \cap T})] \\
&= v(\{i_1, \ldots, i_k\}) \uparrow
\end{aligned}$$

... using a telescoping sum. E.g. sum of first i odd numbers is i^2:

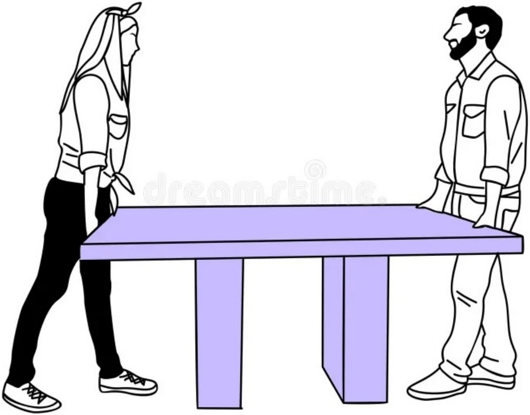
$$1 + 3 + \cdots + (2i - 1) = [1^2 - 0^2] + [2^2 - 1^2] + \cdots + [i^2 - (i-1)^2] = i^2$$

- Why? Supermodularity $\Rightarrow v(B_j \cup i_j) - v(B_j) \geq v(A_j) - v(A_{j-1})$, given $A_{j-1} = \{i_1, \ldots, i_{j-1}\} \subset \{1, \ldots, i_j - 1\} = B_j$

Empty Cores Examples: Not Convex Games

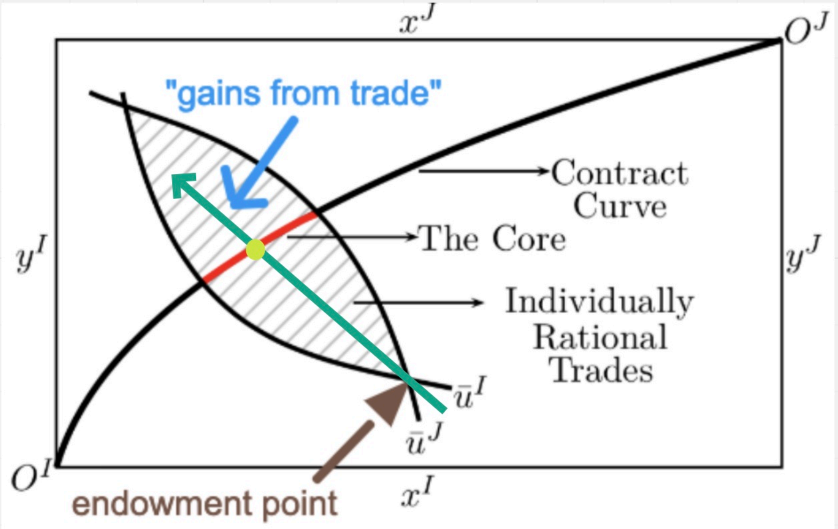
- Carry a table?
- $v(i) = 0,\ v(i,j) = 50 = v(1,2,3)$
- Game is not convex since with $S = (1,2)$ and $T = (2,3)$:

$$0 + 50 = v(2) + v(1,2,3) = v(S \cap T) + v(S \cup T) < v(S) + v(T) = 50 + 50$$



Competitive Equilibrium and the Core

- Now you know why we called it the core earlier on!
- The competitive equilibrium was in the core for $N = 2$



A Stronger First Welfare Theorem Built on the Core

- Coalitions can pool their endowments & trade with each other
- NTU blocking: A coalition trader S blocks an allocation $\mathbf{x}$ if there exists another allocation $\hat{\mathbf{x}}$, feasible from endowments of $i \in S$, with $\hat{\mathbf{x}} \succeq_i \mathbf{x}$ for all $i \in S$ and $\hat{\mathbf{x}} \succ_j \mathbf{x}$ for some $j \in S$.
- Exactly as suggested by the Edgeworth box:

Proposition (Core Welfare Theorem)

If $(\mathbf{x}, \mathbf{p})$ is a competitive equilibrium, then $\mathbf{x}$ is in the core.

- As with First Welfare Theorem, proof is by contradiction
- Let $(\mathbf{x}, \mathbf{p})$ be a competitive equilibrium, but $\mathbf{x} \notin$ core.
- Then some coalition S has a feasible allocation $\widehat{\mathbf{x}}$ with $u^i(\widehat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ for all $i \in S$, strictly so for some $j \in S$.
- Revealed preference $\Rightarrow \mathbf{p} \cdot \widehat{\mathbf{x}}^i \geq \mathbf{p} \cdot \mathbf{x}^i \ \forall i \in S$, and $\mathbf{p} \cdot \widehat{\mathbf{x}}^j > \mathbf{p} \cdot \mathbf{x}^j$.

$\Rightarrow$ $\mathbf{p} \cdot \left(\sum_{i \in S} \widehat{\mathbf{x}}^i\right) > \mathbf{p} \cdot \left(\sum_{i \in S} \mathbf{x}^i\right) = \mathbf{p} \cdot \left(\sum_{i \in S} \mathbf{x}^i\right)$.

- Then $\widehat{\mathbf{x}}$ *is infeasible for the coalition* S: $\sum_{i \in S} \widehat{\mathbf{x}}^i \leq \sum_{i \in S} \mathbf{x}^i$. $\square$

The Shrinking Core of a Market Economy

- We now seek a converse of the last result!
- Debreu and Scarf (1963) proved the reverse of the Core Welfare Theorem holds in large economies
  - This is an amazing endorsement of the competitive model
- Let C_M be the core of the M-clone model.

Proposition (Core Convergence Theorem)

If $\mathbf{x}^* \in C_M$ for all M, then $\mathbf{x}^*$ is a competitive outcome. So the limit of the M-replica cores $\cap_{M=1}^{\infty} C_M$ is a competitive outcome.

The Shrinking Core of a Market Economy: An Example

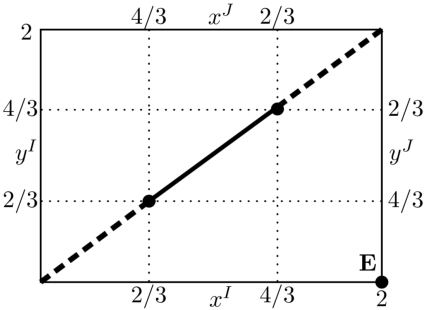
- Agent $k \in \{I, J\}$ with utility function $u^k(x, y) = xy$.
- Endowments diverge: $(\bar{x}^I, \bar{y}^I) = (2, 0)$ and $(\bar{x}^J, \bar{y}^J) = (0, 2)$.
- The core is the diagonal $y^I = x^I$ of the Edgeworth box, since it must be socially efficient
- We now clone each trader: two Irises and two Joes.
- Any allocation with $y^k = x^k$ for $k = I, J$ is still efficient.
  - E.g. $(x^I, y^I) = (0.4, 0.4)$ for Irises and $(x^J, y^J) = (1.6, 1.6)$ for Joes is efficient and IR
  - This allocation yields $u^I = 0.16$ and $u^J = 2.56$.
- $\{I_1, I_2, J_1\}$ blocks with $(x^I, y^I) = (1.2, 0.2), (x^J, y^J) = (1.6, 1.6)$
  - This is feasible: two Irises and one Joe are endowed with $(4, 2)$
  - Irises strictly better off: $u^I(1.2, 0.2) = 0.24 > 0.16 = u^I(0.4, 0.4)$
  - Joe is indifferent. (The excluded Joe is worse off.)

$\Rightarrow$ $(x^I, y^I) = (0.4, 0.4)$ and $(x^J, y^J) = (1.6, 1.6)$ not in the core.

- So what exactly is the core of the 2-replica economy?

Practice Exercise: the Core of the 2-Replica Economy

- Show that the core of the M-Replica Economy is:
  - for $M = 2$, the diagonal $(x^I, y^I) = (a, a)$ for $2/3 < a < 4/3$
  - for $M = 3$, the diagonal $(x^I, y^I) = (a, a)$ for $4/5 < a < 6/5$



Offline: Core of the 2-Replica and 3-Replica Economies

- **2-Replica Economy**
  - The coalition $\{I_1, I_2, J\}$ blocks more allocations.
  - Start at the symmetric efficient allocation $(x^I, y^I) = (a, a)$ and $(x^J, y^J) = (2-a, 2-a)$, with $u^I = a^2$ and $u^J = (2-a)^2$.
  - Reallocate the coalition's $(4, 2)$ endowment so that $(\hat{x}^I, \hat{y}^I) = (1 + a/2, a/2)$ and $(\hat{x}^J, \hat{y}^J) = (2-a, 2-a)$.
  - This blocks the symmetric allocation iff $a < 2/3$:

$$u^I(\hat{x}^I, \hat{y}^I) = \left(\frac{a}{2} + 1\right)\left(\frac{a}{2}\right) > a^2 = u^I(x^I, y^I)$$

  - The core weakly shrinks with each replication, since each adds more coalition constraints.
- **3-Replica Economy**
  - Similarly, show that 3 Irises and 2 Joes block any $a < 4/5$