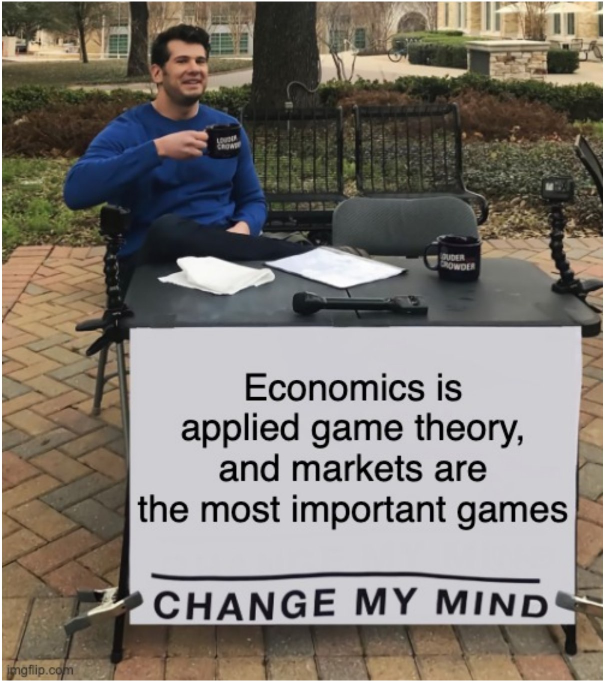
Economic Theory 713A

Economics of Markets

Lones Smith

March 3, 2025

Wisconsin


Economics is
applied game theory,
and markets are
the most important games
CHANGE MY MIND
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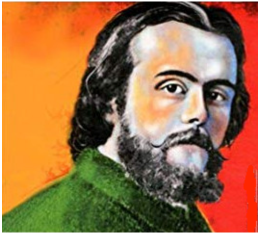
- Leon Walras (1874), *Éléments d'économie politique pure*
  - He first formulates the general equilibrium problem
  - Not translated into English until 1954
  - Econometrica allowed French for decades.



The Walrasian Existence Problem

- Walras formulated the marginal theory of value, based on economic value and not ethical theory of value — extreme laissez-faire:
    - *There is here as well no need to take into account the morality or immorality of the need ...Whether a substance is searched for by a doctor to heal an ill person, or by an assassin to poison his family, this is an important question from other points of view, albeit totally indifferent from ours.*
- He deduced equations for prices and quantities of goods bought and produced, using first order conditions and also Walras Law
- Suggested *tâtonnement* (French for "trial and error")

Eighty Years Pass, and then Arrow and Debreu (1954)

- "Existence of an Equilibrium for a Competitive Economy"
- *"Walras first formulated the state of the economic system at any point of time as the solution of a system of simultaneous equations* ...*Walras did not, however, give any conclusive arguments to show that the equations have a solution"*

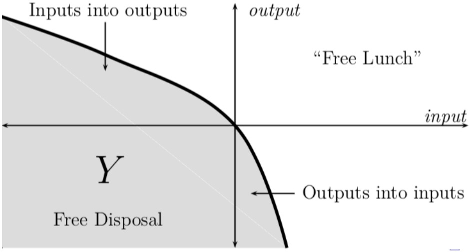
Nash (1951) Inspires Arrow and Debreu (1954)

- "Existence of an Equilibrium for a Competitive Economy"
- Idea: *Professor Nash has formally introduced the notion of an equilibrium point for a game.....*
  - Me: wait, "Professor"???
- *The definition can easily be extended to an abstract economy*
- Goal: introduce an $(m + n + 1)$-player game with
  - m firms maximize profits, and n consumers maximize utility
  - One fictitious Walrasian auctioneer chooses prices to maximize the value of net excess demand $\Rightarrow$ reduce prices of goods in excess supply and raise the prices of goods in excess demand
- They first must extend Nash from a finite game to a continuous action game with quasiconcave payoff functions

Arrow and Debreu's Damn Clever Formulation of Firms

- A firm transforms inputs into outputs
  - A firm is a subset $Y \subset \mathbb{R}^L$, given $L \geq 2$ goods.
  - $y_k \in Y$ is outputs if $y_k > 0$ and input if $y_k < 0$

  $\Rightarrow$ firm profits are $p \cdot y$, the dot product of prices and quantities.
- *Closed convex technology*
  - *no free lunch* $\Rightarrow Y \cap \mathbb{R}^L_+ = \{0\} \Rightarrow 0 \in Y$ (do nothing)
  - *free disposal* $\Rightarrow Y \supset \mathbb{R}^L_-$
  - $Y \subseteq \mathbb{R}^\ell$ is closed and convex (so diminishing returns technologies)
    - Dynamic free lunch is impossible, i.e. NOT
      1 rubber unit $\to$ 2 tires, and 1 tire $\to$ 2 rubber units



Formal Model of the *Competitive* Capitalist Economy

- Each consumer i owns a share $\theta_{ij} \geq 0$ of profits of firm j
- A **competitive equilibrium** of a private ownership economy

$$\left(\{Y^j\}_{j=1}^m, \{X^i, u^i, \bar{\mathbf{x}}^i, \{\theta_{ij}\}_{j=1}^m\}_{i=1}^n\right)$$

is an allocation $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{nL} \times \mathbb{R}^{mL}$ and a price $\mathbf{p} \in \mathbb{R}^L$ so that

- $\forall j$: $\mathbf{y}^j \in Y^j$ maximizes profits, i.e. $\mathbf{p} \cdot \hat{\mathbf{y}}^j \leq \mathbf{p} \cdot \mathbf{y}^j \quad \forall \hat{\mathbf{y}}^j \in Y^j$
- $\forall i$: $\mathbf{x}^i \in X^i$ maximizes utility u^i in the budget set:

$$B^i(\mathbf{p}) = \{\mathbf{x}^i \in X^i : \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \bar{\mathbf{x}}^i + \sum_{j=1}^m \theta_{ij} \mathbf{p} \cdot \mathbf{y}^j\}$$

- Markets clear, namely the excess demand vector is nonpositive:

$$\mathbf{z} = D(p) - \bar{x} - S(p) \equiv \sum_{i=1}^n \mathbf{x}^i - \sum_{i=1}^n \bar{\mathbf{x}}^i - \sum_{j=1}^m \mathbf{y}^j \leq 0$$

and if $z_k < 0$, then $p_k = 0$

Existence Theorem for Competitive Equilibrium

Theorem (Arrow and Debreu, 1954)

Assume consumers $i = 1, \ldots, n$ have continuous, nonsatiated and quasiconcave utility u_i, endowment $\overline{\mathbf{x}}^i \in \mathbb{R}^L_+$, and dividend shares (θ_{ij}). Assume firms $j = 1, \ldots, m$ have closed and convex production technologies. A competitive equilibrium exists.

- This generalizes Nash's existence to games from mixtures over finitely many actions to quasiconcave and continuous payoff functions on a compact convex space
  - Quasiconcavity $\Rightarrow$ pure strategy Nash equilibrium exists
  - Glicksberg '52 extended Nash to linear topological spaces

Corollary (Existence in General Exchange Economies)

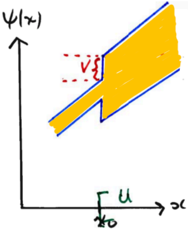
If consumers $i = 1, \ldots, n$ have continuous, nonsatiated and quasiconcave utilities u_i, and endowments $\overline{\mathbf{x}}^i \in \mathbb{R}^L_+$, a competitive equilibrium exists.

- Prove: Existence for the exchange economy follows from considering the special case with no firms ($m = 0$)

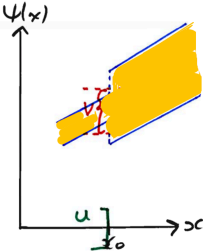
Needs Fancy Fixed Point Math for Set-Valued Functions

Closed Graph Property

- Let $\psi : X \rightrightarrows Y$ be a correspondence, so that $\psi(x) \subset Y$ is a set
- Closed graph property (upper hemicontinuity, UHC) if $\{(x, \psi(x))\}$ contains its limit points: has closed graph in $X \times Y$
    - precludes "implosions" of a correspondence function $\psi(x)$
    - Left plot below is upper hemicontinuous, and right plot is not
    - If ψ is a function, then upper hemicontinuity = continuity
    - Lower hemicontinuity (LHC) precludes "explosions" of ψ
        - Right plot below is lower hemicontinuous, and left plot is not
        - Game theory refinements (e.g. Intuitive Criterion) claim LHC
    - Continuous correspondence = upper + lower hemicontinuous



UHC



LHC

Review: Kakutani Fixed Point Theorem

Theorem (Kakutani Fixed Point Theorem, 1944)

Let ϕ be a correspondence on non-empty, compact, convex $S \subset \mathbb{R}^n$

- *with a closed graph*
- *$\phi(x) \neq \varnothing$ for all $x \in S$.*
- *convex-valued for all $x \in S$*

Then ϕ has a fixed point $x \in \phi(x)$

- "Sir, tell us about the Kakutani FPT." Him: "What's that?"

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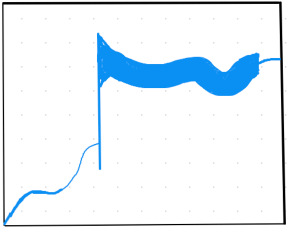
- Kakutani used the von Neumann Approximation lemma to draw a continuous function very close to any closed graph.



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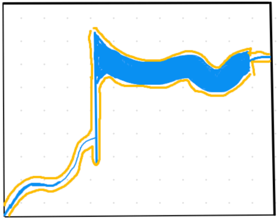
- Kakutani used the von Neumann Approximation lemma:
- Loosely, tighten a tube-sock around the closed graph:



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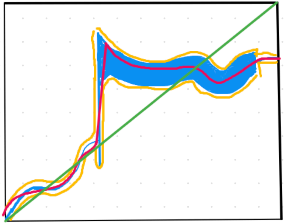
- Kakutani used the von Neumann Approximation lemma:
- Each such function has a fixed point, by Brouwer. Take limits.



Are Action Domains Compact and Convex?

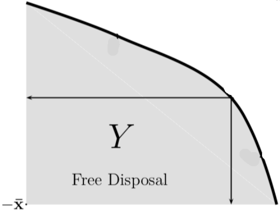
- A competitive equilibrium is a triple (x, y, p) such that:
  - Given p, consumers and firms choose x, y.
  - Given x, y, auctioneer chooses price p (& it clears the market).
- Auctioneer has compact convex action space & finite payoffs
  - Use compact price domain $P = \{p \in \mathbb{R}^L_+ | p_1 + \cdots + p_L = 1\}$.
  - This uses the degree of freedom in Walras Law differently
- Firms and consumers have compact convex action spaces
  - Markets clear $\Rightarrow$ excess demand $\mathbf{z} = D(p) - S(p) - \bar{\mathbf{x}} \leq 0$.

$\Rightarrow$ $S(p) = D(p) - \mathbf{z} - \bar{\mathbf{x}} \geq -\bar{\mathbf{x}}$, since $D(p) - \mathbf{z} = S(p) + \bar{\mathbf{x}} \geq 0$.

$\Rightarrow$ Since Y_j is convex, and $Y_j \cap \mathbb{R}^L_+ = \{0\}$, it is bounded above

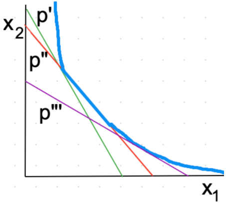
$\Rightarrow$ So every firm's optimization is on a compact domain Y_j.

- Likewise, $D(p) \leq S(p) + \bar{\mathbf{x}}$ is then uniformly bounded above



Arrow-Debreu (1954) Proof Sketch Theorem

- Correspondence $\phi(x_0, y_0, p_0)$ is all best replies in $X \times Y \times P$:
  - all bundles x maximizing utility, given p_0
  - all profit maximizing inputs and outputs y, given p_0
  - all prices p maximizing value of net excess demand, given x_0, y_0
- Continuous u and compact domain $\Rightarrow \phi(x, y, p) \neq \varnothing$
- Theorem of the Maximum $\Rightarrow \phi(x, y, p)$ has a closed graph
- Convex preferences and technologies $\Rightarrow \phi(x, y, p)$ convex

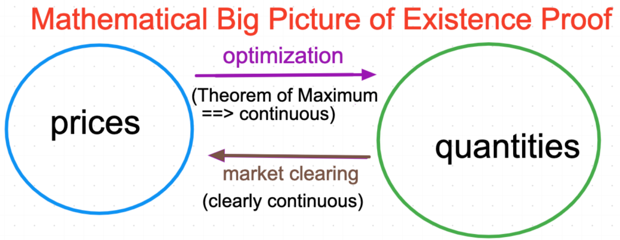


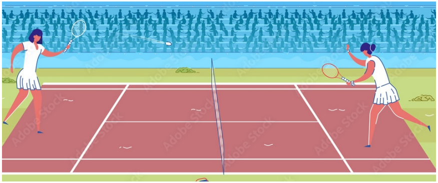
$\Rightarrow$ correspondence $\phi \neq \varnothing$ has a closed graph and convex-valued

- *By Kakutani's Fixed Point Theorem,* $\exists (x, y, p) \in \phi(x, y, p)$
  - Each consumer is optimizing his utility at x, each firm maximizes its profits at y, and markets clear at price p

Insights for Iterative Computer Equilibrium Computation

- Arrow-Debreu proof: $X \times Y \times P \rightarrow X \times Y \times P$ and Kakutani
- Computer: A "tennis game" captures the two maps $X \times Y \rightarrow P$ and $P \rightarrow X \times Y$ (convergence requires prayer)

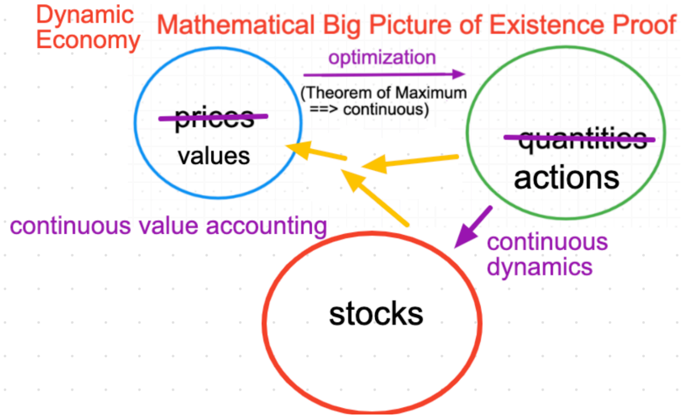




- Walrasian *tâtonnement* usually gives convergence, but there is no general understanding when this works, except say strategic

Equilibrium Computation for Dynamic Search Models

(taught in my Advanced Theory search module)



McKenzie's Slightly Parallel Existence Result

- McKenzie (1954) "On equilibrium in Graham's model of world trade and other competitive systems" also used Kakutani to prove existence
- Lionel Mackenzie founded the Rochester economics department
  - But his was a trade model, and he did not model consumers
  - He did not have the elegant description of firms.
  - Crucially, he did not approach it as a game, but instead as a Kakutani fixed point theorem application
- He did not cite Arrow-Debreu (1954), nor did they cite him!!
- The editor Strotz wrote him in 1953:
  - *"I have given up. Letters have gone to both referees requesting the return of your manuscript to this office right away. I hope to God I can have better luck with the next people. I don't know whether this is a matter of concern to you, but let me assure you that it is my intention not to publish the paper by Arrow and Debreu (which has also been submitted) before the publication of your paper (if both are found acceptable). I think this would only be fair to you.'*
  - Strotz wrote the classic 1955 paper on time consistent preferences. He was President of Northwestern 1970–84.

Socially Efficiency for General Equilibrium with Production

- Given feasible consumption $X \subset \mathbb{R}^{n\ell}$ production: $Y \subset \mathbb{R}^{m\ell}$
- An allocation $(x, y) \in X \times Y$ of a private ownership economy is *socially efficient* if $\nexists(\hat{x}, \hat{y}) \in X \times Y$ such that
  - every consumer i is weakly better off, $u^i(\hat{x}) \geq u^i(x)$
  - some consumer k is strictly better off, $u^k(\hat{x}) > u^k(x)$
  - the allocation $(\hat{x}, \hat{y})$ is feasible (so "markets clear"):

$$\sum_{i=1}^{n} \hat{x}^i - \sum_{i=1}^{n} \bar{\mathbf{x}}^i - \sum_{j=1}^{m} \hat{y}^j \leq 0$$

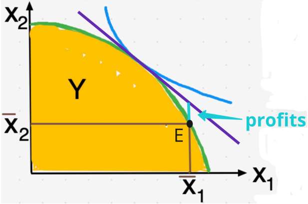
- **Firm profits do not matter here, and do not appear in social welfare functions!**
- Economics welfare analysis is 100% focused on people!
  - We do not treat corporations as people!

1st and 2nd Welfare Theorems with Production

Theorem (Efficiency $\Leftrightarrow$ Competition)

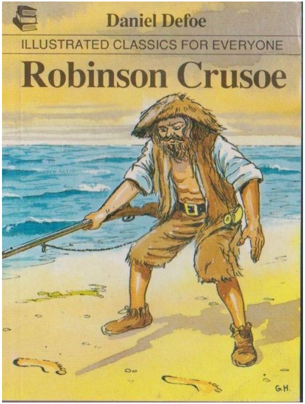
First: If $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ is a competitive equilibrium and preferences are not locally satiated, then $(\mathbf{x}, \mathbf{y})$ is a socially efficient allocation.
Second: Assume monotonic and convex preferences, and closed convex technologies. If $(\mathbf{x}, \mathbf{y})$ is socially efficient, then $(\mathbf{x}, \mathbf{y}, \mathbf{p})$ is a competitive equilibrium, for some prices $\mathbf{p} \in P$, endowments $\bar{\mathbf{x}}$, and ownership shares θ.

- Intuition: Choose the endowment vector $\bar{\mathbf{x}}$ as the Y origin, since it corresponds to the zero production exchange economy
- Open: Modify earlier 2nd welfare theorem proof for production
- Classic Separating Hyperplane logic in a picture



Solvable General Equilibrium: Robinson Crusoe Economies

- macro focuses on a solvable case: $M = 1$ firm, $N = 1$ consumer types
- Karl Marx made this metaphor famous in *Das Kapital*

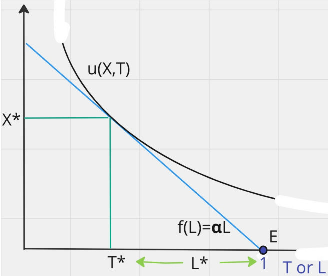


Example: Constant Return to Scale Technology

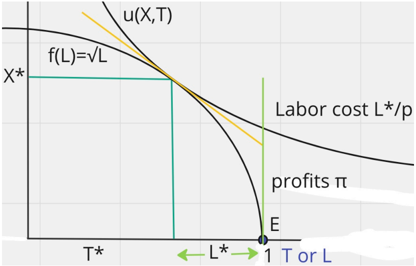
- $L = 2$ goods, produced by $M = 1$ firm, for $N = 1$ consumer
  - Technology: fish or fowl $f(L) = \alpha L$, where L is labor.
  - Preferences: $u(X, T) = X^{\alpha} T^{1-\alpha}$, where T is leisure.
  - Endowment: one unit of time $1 = \bar{T} = L + T$
  - Need not specify firm ownership shares: it earns no profits
- Crusoe Inc. maximizes $pf(L) - L$ iff $p\alpha = 1$, where T is numéraire.
- As endowment income is $\bar{T} = 1$, Cobb Douglas demands are:

$$X = \alpha/p = \alpha^2 \quad \text{and} \quad T = 1 - \alpha$$

- Robinson works $L = \alpha$ hours $\Rightarrow$ Crusoe hires $L = \alpha \Rightarrow X = \alpha^2$
- X market clears (Walras Law)



Example: Diminishing Returns Technology and Profits



- Diminishing returns $\Rightarrow$ profits
- Technology: fish $f(L) = \sqrt{L}$
- Planner: max $u(X, T)$ s.t. $X = \sqrt{L}$
- Competitive Equilibrium (easier)
  - Crusoe Inc. maximizes $p\sqrt{L} - L$.
    - The FOC is $L = p^2/4$
    - $\Rightarrow$ Production is $X = p/2$
    - Profits $pX - L$ are as depicted:

$$\pi = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4} > 0$$

  - Capitalism: Robinson owns Crusoe, Inc.
  - Robinson's income is his endowment value and profits: $1 + \pi$.
    - Leisure demand (using $L = p^2/4$) is

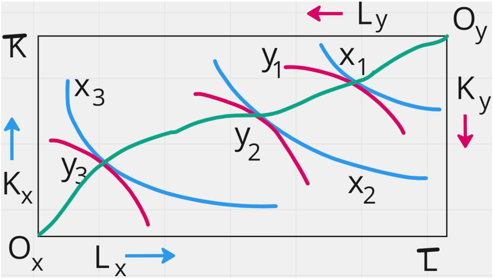
$$T = (1 - \alpha)(1 + p^2/4)$$

    - $T + L = 1 \Rightarrow L^* = \alpha/(2 - \alpha)$ and $T^* = 2(1 - \alpha)/(2 - \alpha)$.
    - $\Rightarrow$ Supply $X^* = \sqrt{\alpha/(2 - \alpha)}$
    - $\Rightarrow$ Competitive price $p = 2X^* = 2\sqrt{\alpha/(2 - \alpha)}$

Multiple Factors of Production or Goods in Competition

- $\nexists$ general solution recipe.
- When in doubt: Find the socially efficient allocation!
- Use the welfare theorems to find wages and prices

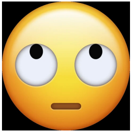
1. Efficient Consumption: Edgeworth Box contract curve
2. Efficient Product Mix: $MRS = MRT$ (taught in public goods)
   - Max utility possibility on production possibility frontier
3. Efficient Production: Find the efficient allocation of factors
   - Every factor is paid the value of its marginal product
     - wage ratio $w_i/w_j = f_i(x)/f_j(x)$ for every industry
     - Every industry has the same the value of its marginal product



Sam Altman: Generative AI Violates Diminishing Returns?

- Theory $\leftrightsquigarrow$ sufficiency: but in our paradigm, we can claim necessity
- AI CEOs might not appreciate the economics of AI: The SOC is locally necessary! At a linear price, marginal product *must be falling*.

> 1. **The intelligence of an AI model roughly equals the log of the resources used to train and run it.** These resources are chiefly training compute, data, and inference compute. It appears that you can spend arbitrary amounts of money and get continuous and predictable gains; the scaling laws that predict this are accurate over many orders of magnitude.
>
> 2. **The cost to use a given level of AI falls about 10x every 12 months, and lower prices lead to much more use.** You can see this in the token cost from GPT-4 in early 2023 to GPT-4o in mid-2024, where the price per token dropped about 150x in that time period. Moore's law changed the world at 2x every 18 months; this is unbelievably stronger.
>
> 3. **The socioeconomic value of linearly increasing intelligence is super-exponential in nature.** A consequence of this is that we see no reason for exponentially increasing investment to stop in the near future.



ChatGPT Question

- See fun 2023 ChatGPT Prelim Question on canvas
  - Will labor suffer losses from AI?