General Social Learning Insights

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April 2, 2025

Informational Inertia

- Standard statistical learning is markovian: the order you see signals is irrelevent. If you helicopter drop into the model, you can proceed just learning the current beliefs
- Social learning is highly path dependent: the action order matters
- *Posterior monotonicity* (PM) asserts:

prior belief rises $\rightarrow$ Joe's posterior belief rises, for a given action by Ike.

- Posterior monotonicity can fail: actions $\Rightarrow$ endogenous signals
  - This is true for statistical learning
  - At higher prior beliefs, Ike takes any action for less favorable private signals $\Rightarrow$ his action less strongly endorses high state.
  - For some signal distributions, this swamps the direct effect of a higher prior public belief.

Informational Inertia

- States $\theta = L, H$ with private belief p with cdfs $F_H(p), F_L(p)$
- Signal log-likelihood ratio $\lambda = \log(dF_H/dF_L)$ has cdf $G_H(\lambda), G_L(\lambda)$ in state $\theta = L, H$.
- This is an equivalent formulation of a signal
- No Introspection Principle:

$$dG_H/dG_L = e^{\lambda}.$$

- Assume three actions: sell, hold, and buy.
- Ike's actions are optimal for *posterior log likelihood ratios* $\lambda_0 + \log[p/(1-p)]$ in $(-\infty, \underline{\lambda})$, $(\underline{\lambda}, \bar{\lambda})$, and $(\bar{\lambda}, \infty)$
- If Ike (with prior λ_0) buys, then Joe's posterior is

$$\lambda_0 + \frac{\int_{\bar{\lambda}-\lambda_0}^{\infty} dG_H(\lambda)}{\int_{\bar{\lambda}-\lambda_0}^{\infty} dG_L(\lambda)} = \lambda_0 + \frac{\int_{\bar{\lambda}-\lambda_0}^{\infty} e^{\lambda} dG_L(\lambda)}{\int_{\bar{\lambda}-\lambda_0}^{\infty} dG_L(\lambda)}$$

- $\exists$ PM if $-\ell + \log\left(\frac{\int_{\ell}^{\infty} e^{\lambda} dF}{\int_{\ell}^{\infty} dF}\right)$ is (strictly) decreasing in ℓ

Private Signals and Private Beliefs

- Two equilikely states $\theta = L, H$
- **Signal Quality Model**: Consider two possible statistically true statements "with chance q, the state is high/low", where the *signal quality* q is distributed over $(0, 1)$ with density γ.
  - $\sigma = \sigma_H$ or σ_L, where $P(\sigma = \sigma_H | H) = q = 1 - P(\sigma = \sigma_H | L)$.
  - If told the state is high, posterior is $q/[q + (1 - q)] = q$
  - If told the state is low, posterior is $1 - q$
  - Ignore atoms (for simplicity). The density of private beliefs p is
    - $f^H(p) = p[\gamma(p) + \gamma(1 - p)]$ in state H
    - $f^L(p) = (1 - p)[\gamma(p) + \gamma(1 - p)]$ in state L
- ***Lemma***: *Under the signal quality structure, private belief distributions are* $F^H(p) \equiv 1 - F^L(1 - p)$ *for all* $p \in (0, 1)$.
- The density of signals for H at p must equal the density for L at strength p for L, and so $1 - p$ for H
- q_n is the private belief of individual n

Random Sampling

- Two actions a and b (eg. 'decline' or 'invest')
- Payoffs $u^H(a) = u^L(a) = 0$, $u^H(b) = 2u$, $u^L(b) = -2$
- Unlike herding literature, entire ordered history is not observed
  - Everyone observes a random *unordered* sample $s \in \mathcal{S}$ of previous action observations
  - Sample size may be random, and sampling weights may also vary over time (uniform, or sample recent past more often)
  - *Aggregates* model: observe whole unordered history
- Sampling is **recursive** if individual $n+1$ samples n with weight π_n, and otherwise individuals $(1, \ldots, n-1)$ as before
  - Stationary recursive sampling is geometric weighting: n samples individual ν with relative weight $\rho^{n-\nu}$, where $\rho > 0$.
  - $\rho \to 0$: only the immediate predecessor is sampled
  - $\rho < 1$: distant past is discounted.
  - $\rho = 1$: *proportional sampling*
  - $\rho > 1$: recent past is undersampled.

Social Beliefs

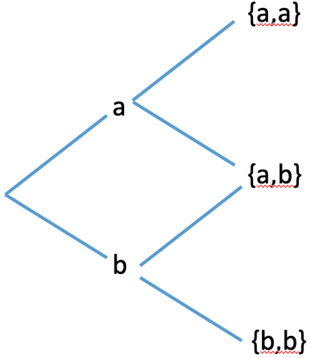
- Every $n = 1, 2, \ldots$ forms a *social belief* q_n that $\theta = H$
- Bayes' rule $\Rightarrow$ posterior belief $r_n = \frac{p_n q_n}{p_n q_n + (1-p_n)(1-q_n)}$
- n chooses action b
  - iff $r_n u \geq (1 - r_n)$
  - iff $p_n \geq (1 - q_n)/[u q_n + (1 - q_n)]$
- How does stochastic process of social beliefs $\langle q_n \rangle$ behave?
  - Is learning complete in the long run? adequate?
  - If not, are there herds? 'proportionate herds'? cycles?

Learning

- Individual n samples individual m with probability $\tau(n, m)$
- Then $\sum_{m=0}^{n-1} \tau(n, m) = 1$ for each n.
- The sampling process *does not over-sample the past* if for all $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $M \geq m$ such that $\tau(n, m) < \varepsilon$ and $\tau(n, 0) < \varepsilon$ for all $n \geq M$.
- By independence of sample sizes, a recursive sampling process (π_n) does not over-sample the past if $\Pi_{n=2}^{\infty}(1 - \pi_n) = 0$.

No More Overturning

- How you arrive at a history is no longer known, but matters
- Consider beliefs after two opposing choices



- We have merged together two information sets with wildly different public beliefs, to create one unified social belief

Two Reasons for Social Learning to Fail

(*a*) If Σ doesn't over-sample the past and private beliefs are unbounded, then learning is complete.

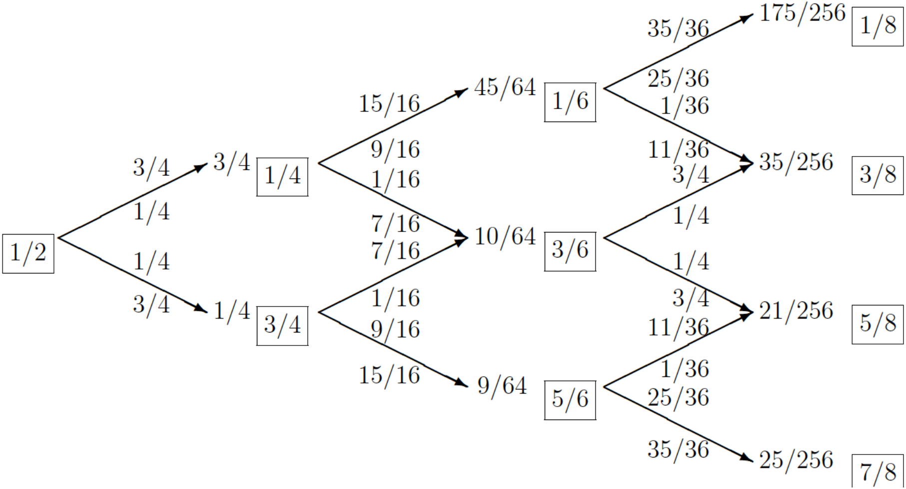
(*b*) Learning is incomplete and payoffs are bounded away from the maximum if Σ over-samples the past.

(*c*) Learning is incomplete and payoffs are bounded away from the maximum for bounded private beliefs and non-empty samples.

Random Sampling

- Not everyone correctly herd with uniform random sampling.
- With unbounded private beliefs, an infinite subsequence of individuals chooses a contrary action.
- **Borel Cantelli Lemma**: If $\sum_{n=1}^{\infty} P(E_n) < \infty$ for events $\{E_n\}$, then the chance that infinitely many events $\{E_n\}$ occur is 0.
  - Proof: $P(\cup_{n=N}^{\infty}(E_n)) \leq \sum_{n=N}^{\infty} P(E_n) \to 0$
- Early individuals have a positive chance of doing anything.
- With random sampling of ≥ 1 predecessors, everyone is a.s. sampled by infinitely many successors,
- Since history become arbitrarily informative, anyone sampling such an individual will eventually choose to follow them
- So, even though the share of individuals taking the right action tends to one, an infinite subsequence takes a suboptimal action with positive probability.

Probabilistic Evolution of Aggregate Observation Model



- Upward transitions are INVEST, and downward ones are NOT
- Tip of arrows are probabilities $\pi_n^H(k)$ (public belief in state L)
- Private belief thresholds $\bar{p}_n(k)$ in the boxes at arrow roots.
- Transition probabilities in states H, L above/below arrows

Are Beliefs a Martingale? (Work with Mingxin Xie)

- After INVEST, the social belief in state L is $\bar{p}_1(1) = 1/4$.
- The expected continuation $E[\bar{p}_2 \,|\, \text{see an investor}]$ is lower:

$$
\begin{aligned}
& \Pr(2 \text{ invests}|1 \text{ invests})\, \bar{p}_2(2) + \Pr(2 \text{ declines}|1 \text{ invests})\, \bar{p}_2(1) \\
= & \left\{[1-\bar{p}_1(1)][1-F^H(\bar{p}_1(1))] + \bar{p}_1(1)[1-F^L(\bar{p}_1(1))]\right\}\bar{p}_2(2) \\
+ & \left\{[1-\bar{p}_1(1)]F^H(\bar{p}_1(1)) + \bar{p}_1(1)F^L(\bar{p}_1(1))\right\}\bar{p}_2(1) \\
= & \tfrac{3}{4}(15/16)+\tfrac{1}{4}(9/16)\}(1/6)+\tfrac{3}{4}(1/16)+\tfrac{1}{4}(7/16)\}(1/2) \\
= & 84/384 \\
< & 1/4
\end{aligned}
$$