Stochastic Calculus: A. An Introduction to the Ito Integral

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Wiener Process via Discrete Time Approximations

- A particle is subject to iid kicks $h\xi_1, h\xi_2, \ldots$ with $P(\xi_i = \pm 1) = \frac{1}{2}$, at the end of length $\delta > 0$ time intervals.

$\Rightarrow$ location of the particle after n kicks is $h \sum_{j=1}^{n} \xi_j$.

- Each $\omega \in \Omega$ fixes a sample path realization, so $\xi_j = \xi_j(\omega)$
- At time t, its location is $S_t(\omega) = h \sum_{j=1}^{\lfloor \frac{t}{\delta} \rfloor} \xi_j(\omega)$.
  - $ES_t = 0$ and $Var(S_t) = h^2 \cdot \lfloor \frac{t}{\delta} \rfloor \approx \frac{h^2}{\delta} t$.
- Fix $\sigma^2 > 0$. Take the limit as $h, \delta \to 0$ with $h^2 = \sigma^2 \delta$

$\Rightarrow$ With $\delta = \frac{1}{n}$ and $h_n = \frac{\sigma}{\sqrt{n}}$, we have $S_t(\omega) = \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{nt} \xi_j(\omega)$.

- A "functional" central limit theorem yields $S_t(\omega) \longrightarrow \{W_t\}$ for $n \longrightarrow \infty$ — a Brownian motion, or a Wiener process

Basic Properties of Wiener Process

- For a standard Wiener process ($\sigma = 1$, and $W_0 = 0$):
  1. $\{W_t\}$ has independent increments:

     $$0 \leq t_1 \leq t_2 \leq \ldots \leq t_k = T$$

     then $W_{t_1} - W_{t_0}, \ldots (W_{t_k} - W_{t_{k-1}})$ are independent.
     - Proof: Terms reflect disjoint summands.
  2. Stationarity: every $W_{t+s} - W_s \sim N(0, t)$.
     - Proof: $W_{t+s} = W_s + [W_{t+s} - W_s]$
  3. $W_0 = 0$ and W_t is continuous
- Kolmogorov: Such a stochastic process exists
- Billingsley, "Convergence of Probability Measures" (1968): the above discrete time random walks converge
- Gentle: Avinish Dixit (1993) "The Art of Smooth Pasting"

Wiener Process

- Covariance function:

$$\mathrm{cov}(W_t, W_s) = E(W_t, W_s) - E(W_t)E(W_s) = \min\{s, t\}$$

- Proof: If $t < s$, then using the fact of disjoint sums yields

$$\begin{aligned} E(W_t, W_s) &= E(W_t(W_t + W_s - W_t)) \\ &= E(W_t, W_t) + E((W_t - W_0)(W_s - W_t)) = t \end{aligned}$$

- The transition probability density of W_t is given by

$$Pr(W_t \in F | W_0 = x_0) = \int_{x \in F} P(x, t | x_0) dx$$

where $p(x, t|x_0) = e^{\frac{-(x-x_0)^2}{2t}} / \sqrt{2\pi t}$

- $\{W_t\}$ is a Markov process

$$Pr(W_t \in F | W_r, t_0 \le r \le s) = Pr(W_t \in F | W_s)$$

- *Given the present, the past and future are independent.*
- $\{W_t\}$ is a martingale. For any $t > s$

$$E(W_t | W_r, t_0 \le r \le s, W = x) = x$$

Wiener Process and Discrete Time Approximations

- $\{W_t\}$ is a time homogenous process, which implies
$$Pr(W_{t+s} \in F | W_s = x_0) = Pr(W_t \in F | W_0 = x_0)$$
- $\{W_t\}$ has zero long run velocity: $Pr(\lim_{t \longrightarrow \infty} \frac{W_t}{t} = 0) = 1$
  - Proof: $W_t - W_0 \sim N(0, t) \Longrightarrow \frac{W_t}{t} \sim N(0, \frac{1}{t})$
- $\{W_t\}$ is everywhere non-differentiable
  - Proof: $\frac{W_{t+h} - W_t}{h} \sim N(0, \frac{\sigma^2}{h})$ diverges as $h \longrightarrow 0$

  $\Rightarrow$ $Pr(\frac{W_{t+h} - W_t}{h} \in F) \longrightarrow 0$ for all bounded interval F.

  $\Rightarrow$ cannot converge with a positive probability to a r.v.
  - In the short run, *volatility dominates the trend*
- $\{W_t\}$ has unbounded expected variation on any subinterval.
  - Proof: For any $s = t_0 < t_1 < \cdots < t_n = t$

$$\begin{aligned}
\sum |W_{t_k} - W_{t_{k-1}}|^2 &\leq \max_k |W_{t_k} - W_{t_{k-1}}| \sum_{k=1}^{n} |W_{t_k} - W_{t_{k-1}}| \\
&= (\max_k |W_{t_k} - W_{t_{k-1}}|) \cdot \textstyle\sum_{k=1}^{n} |W_{t_k} - W_{t_{k-1}}|
\end{aligned}$$

  - The LHS has expectation $\sum (t_k - t_{k-1}) = t - s > 0$.
  - The first factor on RHS vanishes vanishes in partition fineness

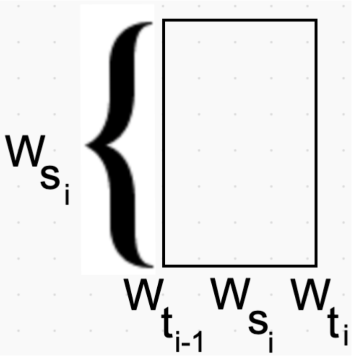
  $\Rightarrow$ total variation explodes in the partition fineness

Stochastic Integration

- First, $\int_{t_0}^{t} 1 dW_s = W_t - W_{t_0}$.
  - Proof: Every Riemann-Stieltjes sum like

$$RS_n = \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})$$

    equals $W_t - W_{t_0}$, regardless of choice $t_0 \leq t_1 \leq \cdots \leq t_n = t$.
- But for $\int_{t_0}^{t} W_s dW_s$, the Riemann-Stieltjes summands look like



Stochastic Integration: What is $\int_{t_0}^t W_s dW_s$?

- Riemann-Stieltjes sums terms like

$$\begin{aligned}
& W_{s_i}(W_{t_i} - W_{t_{i-1}}) \\
= \; & W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) + (W_{s_i} - W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\
= \; & \tfrac{1}{2}(W_{t_i} + W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) - \tfrac{1}{2}(W_{t_i} - W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\
& +(W_{s_i} - W_{t_{i-1}})(W_{s_i} - W_{t_{i-1}}) + (W_{t_i} - W_{s_i})(W_{s_i} - W_{t_{i-1}})
\end{aligned}$$

- The sum over i is $\frac{1}{2}(W_t^2 - W_{t_0}^2) + A_n + B_n + C_n$, where

$$\begin{aligned}
A_n &= -\tfrac{1}{2}\textstyle\sum_{i=1}^n (W_{t_i} - W_{t_{i-1}})^2 \\
B_n &= \textstyle\sum_{i=1}^n (W_{s_i} - W_{t_{i-1}})^2 \\
C_n &= \textstyle\sum_{i=1}^n (W_{t_i} - W_{s_i})(W_{s_i} - W_{t_{i-1}})
\end{aligned}$$

- *Each term has limit zero variance*. Eg. $Var(W_t^2) = 2t^2 \Rightarrow$

  $var(A_n) = \frac{1}{2} 2 \sum_{i=1}^n (t_i - t_{i-1})^2 \leq (t - t_0) \max_i |t_i - t_{i-1}| \to 0$
- $E[A_n] = -\frac{1}{2}[\sum_{i=1}^n (W_{t_i} - W_{t_{i-1}})^2] = -\frac{1}{2}(t - t_0)$ (Gaussian)
- $E[B_n] = \sum_{i=1}^n (W_{s_i} - W_{t_{i-1}})^2 = \sum (s_i - t_{i-1})$ (Gaussian)
- $E[C_n] = \sum_{i=1}^n (W_{t_i} - W_{s_i})(W_{s_i} - W_{t_{i-1}}) = 0$ (independence)

Stochastic Integration: What is $\int_{t_0}^{t} W_s dW_s$?

- The sums RS_n converges in quadratic mean to the r.v.

$$\tfrac{1}{2}(W_t^2 - W_{t_0}^2) + (\theta - \tfrac{1}{2})(t - t_0)$$

- *It matters how we choose our intermediate points $\{s_i\}$.*
- Let $s_i = (1-\theta)t_{i-1} + \theta t_i$, where $0 \le \theta \le 1$,
- Then $\sum_{i=1}^{n}(s_i - t_{i-1}) = \theta \sum_{i=1}^{n}(t_i - t_{i-1}) = \theta(t - t_0)$.
- Stratonovich Integral uses $\theta = \frac{1}{2}$:

$$(S)\int_{t_0}^{t} W_s dW_s = \tfrac{1}{2}(W_t^2 - W_{t_0}^2)$$

- Ito integral uses $\theta = 0$:

$$\int_{t_0}^{t} W_s dW_s = \tfrac{1}{2}(W_t^2 - W_{t_0}^2) - (t - t_0)$$

- Economists use Ito integrals, since control decisions cannot depend not on yet-to-be realized noise.
- Bonus: $x_t = \frac{1}{2}{W_t}^2 + (\theta - \frac{1}{2})t$ is a martingale iff $\theta = 0$

Norbert Wiener (1923) and Ito (1944)

- Norbert Wiener (1923)
  - 1948, Cybernetics: Or Control and Communication in the Animal and the Machine
- Kiyosi Itô