Stochastic Calculus: B. Ito's Lemma, Boundary Value Problems, and Filtering

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Stochastic Integration and Ito's Lemma

- A stochastic differential equation

$$\boxed{dx_t = \mu(x,t)dt + \sigma(x,t)dW_t} \qquad (\bigstar)$$

  is just notation for a stochastic integral:

$$\boxed{x_t = x_0 + \int_0^t \mu(x,s)ds + \int_0^t \sigma(x,s)dW_s}$$

- Ito's Lemma (The Probabilistic Chain Rule): Assume $(\bigstar)$.
  If $Y_t = g(t, x_T)$ *where* $g(t,x)$ *is* C^2*, then*

$$\boxed{dY_t = g_t dt + g_x dX + \frac{1}{2} g_{xx}\sigma^2(x,t)dt}$$

- Hint: Arrow-Pratt risk aversion coefficient motivation
- Example: We just proved that $WdW = \frac{1}{2}d(W^2) - \frac{1}{2}dt$.

$\Rightarrow$ Ito's Lemma for $g(x) = \frac{1}{2}x^2$, that $d(\frac{1}{2}W^2) = WdW + \frac{1}{2} \cdot dt$

Heuristic Proof of Ito's Lemma

- Since $dW \sim N(0, dt)$, we have $E[dW^2] = dt$
- We earlier illustrated the multiplication table

X	dt	dW
dt	0	0
dW	0	dt

  - Try to prove $(dt)(dW) = O((dt)^{2/3})$ from the properties of the Gaussian distribution
- By Taylor's Theorem, if $dX = \mu dt + \sigma dW$, then

$$\begin{aligned} dg &= g_x dX + g_t dt + \tfrac{1}{2}[g_{xx}(dX)^2 + 2g_{xt}(dX)(dt) + g_{tt}(dt)^2] \\ &= g_x dX + g_t dt + \tfrac{1}{2} g_{xx}\sigma^2 dt \end{aligned}$$

Geometric Brownian Motion

- If $dX = \mu dt + \sigma dW$ find the SDE of $Y_t = e_t^X$
- Guess $g(x) = e^x$. Since $g''(x) = e^x$, Ito implies

$$dY_t = e^{X_t}[dX_t + \tfrac{1}{2}(dX_t)^2] \Rightarrow \frac{dY_t}{Y_t} = (\mu + \tfrac{1}{2}\sigma^2)dt + \sigma dW_t \quad (\dagger)$$

$\Rightarrow$ Greater volatility σ^2 raises the drift of (Y_t).

- In practice, we are asked to "solve" equations like $(\dagger)$

$\Rightarrow$ Solution: We unwrap Y_t into $Y_t = e_t^X$, for which we have a closed formula: $X_t = \mu t + \sigma W_t$, where $W_t \sim N(0, 1)$
  - "Reversing" Ito's rule — integrating — is an art

Boundary Value Problems

Theorem

Let T be the random hitting time of (X_t) from $[a, b]$, where

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

Posit flow payoff $\pi(x)$, terminal payoffs $\phi(a)$, $\phi(b)$, interest rate β:

$$V(x) = E\left[\int_0^T e^{-\beta t}\pi(X_t)dt + e^{-\beta T}\phi(X_T)\middle|X_0 = x\right]$$

Then V solves $V(a) = \phi(a)$ and $V(b) = \phi(b)$ and

$$\beta V(x) = \pi(x) + [\mu(x)V'(x) + \tfrac{1}{2}\sigma(x)^2 V''(x)]$$

In other words: **asset return** $=$ **dividend** $+$ **E(capital gain)**].
Proof: Advance $\varepsilon > 0$ in time, and let $o(\varepsilon)/\varepsilon$ vanish in ε.

$$\begin{aligned} V(x) &= \pi(x)\varepsilon + e^{-\beta\varepsilon}E[V(x + \Delta X)] + o(\varepsilon) \\ &= \pi(x)\varepsilon + (1 - \beta\varepsilon)[V(x) + (\Delta X)V'(x) + \tfrac{1}{2}(\Delta X)^2 V''(x)] + o(\varepsilon) \end{aligned}$$

Subtract $V(x)$, then divide by ε, and let $\varepsilon \to 0$, with $o(\varepsilon)/\varepsilon \to 0$.

Example: Present Value of Utility from an Asset

- Geometric Brownian motion: $dX_t = X_t(\mu dt + \sigma dW)$. $(\bigstar)$
- Present value $V(x) = E_x(\int_0^\infty e^{-rt}(-X_t^{-\gamma})dt)$

$$\Rightarrow rV(x) = -x^{-\gamma} + \mu x V'(x) + \tfrac{1}{2}\sigma^2 x^2 V''(x). \qquad (\ddagger)$$

- $V_h(x) = x^\lambda$ is general solution of the homogeneous equation

$$rV_h(x) = \mu x V_h'(x) + \tfrac{1}{2}\sigma^2 x^2 V_h''(x)$$

$\Rightarrow$ $(1/2)\sigma^2\lambda(\lambda-1) + \mu\lambda - r = 0$

$\Rightarrow$ $V_h(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$ for roots $\lambda_1 < 0 < \lambda_2$

- $V_h(\infty) = 0 \Rightarrow c_2 = 0$.

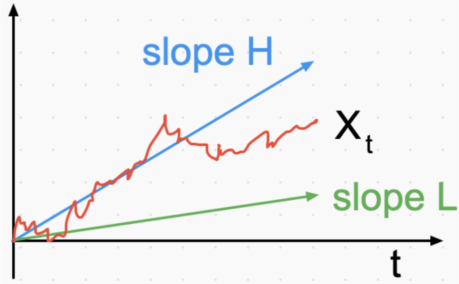
- $V_p(x) = c_0 x^{-\gamma}$ is the particular solution of $(\ddagger)$

$$rc_0 = -1 - \mu\gamma c_0 + c_0\gamma(\gamma+1)\sigma^2/2 \quad \Rightarrow \quad c_0 = \frac{2}{\sigma^2\gamma^2 + (\sigma^2 - \mu)\gamma - r}$$

- $V(x) = c_0 x^{-\gamma} + c_1 x^{\lambda_1}$ is the general solution
- Put $Y_t = \alpha X_t$ in $(\bigstar)$ to prove $V(\alpha x) \equiv \alpha^{-\gamma} V(x) \Rightarrow c_1 = 0$

$$\boxed{V(x) = \frac{2x^{-\gamma}}{\sigma^2\gamma^2 + (\sigma^2 - \mu)\gamma - r}}$$

Binary State Bayes Rule



- Learning about a fixed *state*, low or high ($\theta = L$ or $\theta = H$).
- Observation process $dX_t^\theta = \theta dt + \sigma dW_t$ for $\theta \in \{L, H\}$.

Lemma

The conditional chance p_t that $\theta = H$ obeys

$$dp_t = \frac{H-L}{\sigma} p_t(1-p_t)d\bar{W}_t$$

where $d\bar{W}_t$ is the expected Wiener increment at time t, i.e. $dX_t = M(p)dt + \sigma d\bar{W}_t$, where $M(p) = pH + (1-p)L$ is the drift.

- The law of motion is a martingale (no drift dt term)

Gentle Proof of the Binary State Bayes Rule

- Observation process (Y_t) has drift $\vartheta = h = H/\sigma$ or $\vartheta = \ell = L/\sigma$.
- Increment $\Delta Y_t = \vartheta \Delta t + \Delta \bar{W} \sim N(\vartheta \Delta t, \Delta t)$ on $[t, t + \Delta t]$,
- An increment Δy has probability density $f_\vartheta(\Delta y)$ in state ϑ

$$\frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{(\Delta y - \vartheta\Delta t)^2}{2\Delta t}\right) = \frac{e^{-(\Delta y)^2/2\Delta t}}{\sqrt{2\pi\Delta t}} \exp\left(\vartheta\Delta y - \vartheta^2\Delta t/2\right) + o(\Delta t)$$

$$\begin{aligned}
\Rightarrow f_\vartheta(\Delta y) &\equiv c \cdot \exp\left(\vartheta \Delta y - \vartheta^2 \Delta t/2\right) \quad (c \text{ constant in } \vartheta) \\
&= c \cdot [1 + \vartheta\Delta y - \vartheta^2\Delta t/2 + \frac{1}{2}(\vartheta\Delta y - \vartheta^2\Delta t/2)^2] + o((\Delta t)^2) \\
&= c \cdot [1 + \vartheta \Delta y] + o(\Delta t)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \Delta p_t = p_{t+\Delta t} - p_t &= \frac{p_t f_h(\Delta y)}{p_t f_h(\Delta y) + [1 - p_t] f_\ell(\Delta y)} - p_t + o(\Delta t) \\
&= \frac{p_t(1 - p_t)[f_h(\Delta y) - f_\ell(\Delta y)]}{p_t f_h(\Delta y) + [1 - p_t] f_\ell(\Delta y)} + o(\Delta t) \\
&= \frac{p_t(1 - p_t)(h - \ell)\Delta y}{1 + m(p_t)\Delta y} + o(\Delta t) \\
&\approx p_t(1 - p_t)(h - \ell)\Delta \bar{W} + o(\Delta t)
\end{aligned}$$

The Belief Process in the Poisson State Switching

- Assume the state θ switches periodically:
- State θ switches $L \to H$ at rate b, and $H \to L$ at rate a
  - So in $[t, t + dt]$, state H changes to L with chance adt
  - Then $dp_t = [(1 - p_t)a + p_t b]dt$
  - So $dp_t \gtrless 0$ as $p_t \gtrless \frac{a}{a+b}$