Stochastic Calculus: III. Smooth Pasting and the Optimal Level of Experimentation

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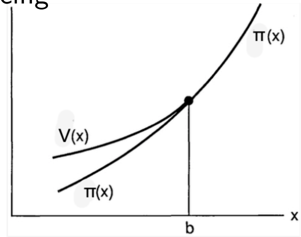
Optimal Stopping: Smooth Pasting (Samuelson, 1965)

- "A Rational Theory of Warrant Pricing"
- SDE $dX_t = \mu dt + \sigma dW$.
- Terminal payoff $\pi(x)$
- Optimal value $V(x) \geq \pi(x)$
- Stopping is best for $X_t > b$
- Waiting is best for $X_t < b$
- Value Matching: $\pi(b) = V(b)$.
- Smooth Pasting: $\pi'(b) = V'(b)$



- Proof: We use a random walk approximation, namely, $\Delta X = \mu \Delta t \pm \sigma\sqrt{\Delta t}$ with both equilikely
- If $\pi'(b) > V'(b)$, then Δt delay beats stopping at $X_t = b$:

$$
\begin{aligned}
E[V(t+\Delta t)] = & \quad \tfrac{1}{2}\pi(b+\mu\Delta t+\sigma\sqrt{\Delta t}) + \tfrac{1}{2}V(b+\mu\Delta t-\sigma\sqrt{\Delta t}) \\
\approx & \quad \pi(b) + \tfrac{1}{2}\pi'(b+)\sigma\sqrt{\Delta t} - \tfrac{1}{2}V'(b-)\sigma\sqrt{\Delta t} + O(\Delta t) > V(b)
\end{aligned}
$$

by value matching and two Taylor approximations about b

- Idea: volatility is so tremendous that smooth pasting is needed
- Economic big idea: the option value of delay!

Finance Application of Smooth Pasting

- Sell asset of value X_T at time T for transaction cost $c > 0$
- Geometric Brownian motion: $dX_t = X_t(\mu dt + \sigma dW_t)$.
- Optimal Stopping rule: stop when $X_t = \bar{x} > c > 0$
- Skip asset value BVP derivation for value of stopping at $\bar{x}$

$$V(x) = (\bar{x} - c)\left(\frac{x}{\bar{x}}\right)^{\lambda_2} \quad \text{where} \quad \lambda_2 = \frac{\sigma^2 - 2\mu + \sqrt{(\sigma^2 - 2\mu)^2 + \delta r \sigma^2}}{2\sigma^2}$$

- Optimization in $\bar{x}$ yields the first order condition

$$(\bar{x} - c)\lambda_2 = \bar{x} \qquad \Rightarrow \qquad \bar{x} = \frac{c\lambda_2}{\lambda_2 - 1}$$

- For the terminal value $\pi(x) = x - c$, smooth pasting yields:

$$V'(x)|_{x=\bar{x}} = \pi'(x)|_{x=\bar{x}}$$

- Then

$$\lambda_2(\bar{x} - c)\bar{x}^{-1} = 1 \Rightarrow \bar{x} = \frac{c\lambda_2}{\lambda_2 - 1}$$

- We do not need smooth pasting here, but confirm it works
- See *The Art of Smooth Pasting* by Dixit

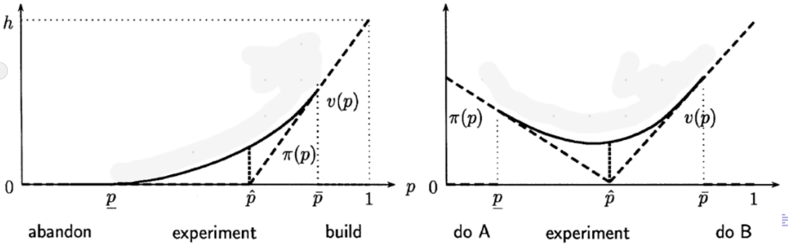
Chernoff's (1972) Experimentation Model

- One-shot payoffs π_a^θ from action $a \in \{A, B\}$ in state $\theta \in \{0, 1\}$.

$\Rightarrow$ expected payoffs from action a are $\pi_a(p) = p\pi_a^1 + (1-p)\pi_a^0$.

- Action A is best in state 0 and action B is best in state 1.
- One-shot payoff is $\bar{\pi}(p) = \max(\pi_A^0(p), \pi_B^1(p))$, equality at $\hat{p}$.
- Chernoff, 1972: "Sequential Analysis and Optimal Design"
  - Initial experimentation phase has flow cost $c > 0$ per unit time
  - Sam sees sample path of $dX_t = \theta dt + \sigma dW_t$

  $\Rightarrow$ Bayes rule: $dP_t = \alpha P_t(1-P_t)d\bar{W}_t$, for an Ito process $\bar{W}_t$.
  - Optimal solution is a stopping time T: continue until the posterior P_T exits $[\underline{p}, \bar{p}]$, where $\underline{p} < \hat{p} < \bar{p}$.
  - Homework: Use value matching & smooth pasting to find $\underline{p}, \bar{p}$
  - Hint: change variables to $\lambda_t = \log[P_t/(1-P_t)]$



Optimal Level of Experimentation (Moscarini-Smith, '01)

- Chernoff had an extensive margin. Add an intensive margin!
- Sam is impatient and experiments at variable intensity.
- Experimentation level n has flow cost $c(n)$, with $c', c'' > 0$.
  - With a constant marginal cost of experimentation, Sam intuitively stops in an "instant", when P_t hits $\underline{p}$ or $\bar{p}$.
- Stats 101: Sample size $n \Rightarrow$ Sample mean variance scales by $\frac{1}{n}$
- Experimentation level N_t at time t: $dX_t = \theta dt + \frac{\sigma}{\sqrt{N_t}} dW_t$

$\Rightarrow$ Belief volatility likewise scales: $dP_t = \alpha\sqrt{N} P_t(1 - P_t) d\bar{W}_t$.

$$V(p_0) = \max_{T, N_t} E\left[\int_0^T -c(n_t)e^{-rt}dt + e^{-rT}\bar{\pi}(P_T)\right] \text{s.t.} dP_t = \alpha\sqrt{N} P_t(1 - P_t) d\bar{W}_t$$

Optimal Level of Experimentation (Moscarini-Smith, '01)

- Optimal stopping solved as before
- Since $E[dP_t] = 0$, the optimal control Bellman equation is

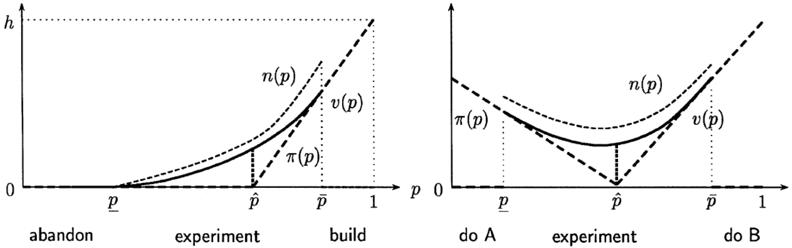
$$rV(p) = \max_{n \geq 0} -c(n) + \frac{1}{2} np^2(1-p)^2 V''(p) \Rightarrow \text{FOC } c'(n) = \frac{1}{2} p^2(1-p)^2 V''(p)$$

$\Rightarrow$ $rV(p) = -c(n) + nc'(n) = g(n)$, where $g'(n) = nc''(n) > 0$.

$\Rightarrow$ Return rV is the producer surplus of experimentation $g(n)$.

- Optimal experimentation level $N_t = g^{-1}(rV(P_t)) \uparrow V(P_t)$
- We never explicitly solve the experimentation level!
- (N_t) *drifts up/down if* $g^{-1}(rV(p))$ *is strictly convex/concave*
- So $c(n) = n^2 \Rightarrow g(n) = n^2 \Rightarrow n(p) = g^{-1}(rV(p)) = \sqrt{rV(p)}$

Backward-Bending Demand for Experimentation



- Classic case: build if $\theta = 1$, abandon if $\theta = 0 \Rightarrow n'(p) \uparrow p$.
- General case with a $\vee$-shaped terminal payoff frontier $\Rightarrow$ U-shaped level, i.e. *experimentation is highest near the end!*
- The optimal experimentation level increases in the interest rate r near the stopping thresholds $\underline{p}$ and $\bar{p}$

$\Rightarrow$ The demand for experimentation is "backward-bending" as a function of interest rate near the end

  - Why? Experimenter rushes to finish sooner.

- Raj Chetty (2007) derived the backward-bending demand for investment in a 2-period model ["Interest Rates, Irreversibility, and Backward-Bending Investment", *REStud*]

Experimentation Regimes and Super Smooth Pasting

- Skill X_t obeys $dX = \mu X dt + X dW_t$
- Win prize $\pi > 0$ if skill hits $X = 1$
- Investment levels $\mu = L < H$ have flow cost $\lambda < h$
- Payoffs are discounted at the interest rate $r > 0$
- Investment rises in value $\Rightarrow$ threshold θ s.t. $\mu = L$ iff $x < \theta$

$$rV_L(x) = -\lambda + LxV_L'(x) + \tfrac{1}{2}x^2 V_L''(x) \quad \Rightarrow \quad V_L(x) = -\frac{\lambda}{r} + a_1 x^{\alpha_1} + a_2 x^{\alpha_2}$$

- Since $\alpha_1 < 0 < \alpha_2$, we have $V_L(0) < \infty$ iff $a_1 = 0$.

$$rV_H(x) = -h + LxV_H'(x) + \tfrac{1}{2}x^2 V_H''(x) \quad \Rightarrow \quad V_H(x) = -\frac{h}{r} + b_1 x^{\beta_1} + b_2 x^{\beta_2}$$

- Unknowns are θ, a_2, b_1, b_2

Experimentation Regimes and Super Smooth Pasting

- Equations are:
  1. boundary condition: $V_H(1) = \pi$
  2. value matching condition: $V_L(s) = V_H(s)$
  3. smooth pasting condition: $V'_L(s) = V'_H(s)$
  4. supersmooth pasting condition: $V''_L(s) = V''_H(s)$.
- Bernard Dumas (1991), "Super contact and related optimality conditions", *Journal of Economic Dynamics and Control* (now defunct)