Insights on Monotone Methods in Economics You Must Know

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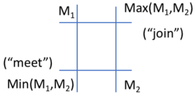
1. Monotone Comparative Statics: Basics Review
   - Lattices and Supermodularity
   - Comparative Statics
   - Stochastic Dominance
   - Information
   - Monotone Likelihood Ratio Property (MLRP)

2. Monotone Comparative Statics Under Uncertainty
   - Logsupermodularity
   - Logconcavity and LSPM
   - Upcrossing Preservation
   - Interval Dominance Order
   - Total Positivity and the Variation Diminishing Property

Join, Meet, Lattice

- A **poset** is a set X and a partial order $\succeq$
- The **join** $x \vee x'$ is the supremum of x, x'

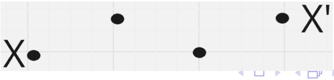


- The **meet** $x \wedge x'$ the infimum of x and x'
- A **lattice** is a poset that contains all meets and joins
- We restrict to Euclidean lattices $X \subset \mathbb{R}^n$, where

$$\mathbf{x} \vee \mathbf{x}' = (\max\{x_1, x_1'\}, ..., \max\{x_N, x_N'\})$$
$$\mathbf{x} \wedge \mathbf{x}' = (\min\{x_1, x_1'\}, ..., \min\{x_N, x_N'\})$$

- **Strong Set Order (SSO)**, denoted $\sqsupseteq$
- $X' \sqsupseteq X$ if for all $x \in X, x' \in X'$, $x \vee x' \in X'$ & $x \wedge x' \in X$.

- Prove $X' \sqsupseteq X$ fails here:



Exotic Lattices

- What do we call the meet and join of two numbers for these posets:
  1. integers ordered by $b \succeq a$ if a is a factor of b
  2. subsets of a finite set ordered by $B \succeq A$ if a is a subset of b
  3. partitions a set S into disjoint, non-empty subets ordered by: $B \succeq A$ if B is a coarsening of A
     - E.g. For $S = \{1, 2, 3\}$, the partition $\{\{1, 2\}, \{3\}\} \preceq \{\{1, 2, 3\}\}$, but $\{\{1, 2\}, \{3\}\}$ is incomparable to $\{\{1, 3\}, \{2\}\}$.

- $F : X \to \mathbb{R}$ is **supermodular** (SPM) if for all $x, x' \in X$

$$F(x \wedge x') + F(x \vee x') \geq F(x) + F(x')$$

- Fact: A function on a totally ordered set (*chain*) is SPM
- If $F(x, \theta)$ is SPM, then F has **increasing differences** (ID) in (x, θ) if $F(x_2, \theta) - F(x_1, \theta)$ increases in θ, for all $x_2 > x_1$
- If $F : \mathbb{R}^n \to \mathbb{R}$ is C^2, then F is SPM iff $\frac{\partial^2 F}{\partial x_i \partial x_j} \geq 0$ for all x
- Addition: If $F, G : X \to \mathbb{R}$ are SPM, then $F + G$ is SPM

Lemma (Maximization Preserves SPM)

F SPM on the lattice $X \times Y \Rightarrow G(x) = \sup_y F(x, y)$ SPM on X.

- *Proof:* Let $y, y' \in Y$ and $x, x' \in X$. Since F is SPM:

$$\begin{aligned} F(x', y') + F(x, y) &\leq F(x \vee x', y' \vee y) + F(x \wedge x', y' \wedge y) \\ &\leq G(x' \vee x) + G(x' \wedge x) \end{aligned}$$

- So $G(x' \vee x) + G(x' \wedge x)$ is an upper bound for the LHS.
- Maximizing the left side over all y, y', we get:

$$G(x') + G(x) \leq G(x' \vee x) + G(x' \wedge x) \qquad \square$$

Comparative Statics

- Let $\mathbf{X}^*(\theta)$ be the set of solutions to the problem

$$\max_{\mathbf{x}\in X} F(\mathbf{x},\theta)$$

- **Topkis Theorem (1978):** *Let X be a lattice, and Θ a poset. If $F : X \times \Theta \to \mathbb{R}$ has ID in (x,θ) and is SPM in x, then $X^*(\theta)$ is monotone in the SSO.*
- *Proof:* Let $\theta' \succ \theta''$ and $x' \in X^*(\theta')$ and $x'' \in X^*(\theta'')$.

$$\begin{aligned}
0 &\geq F(x' \vee x'', \theta') - F(x', \theta') && \text{by } x' \in X^*(\theta') \\
&\geq F(x' \vee x'', \theta'') - F(x', \theta'') && \text{by ID in } (x,\theta) \\
&\geq F(x'', \theta'') - F(x' \wedge x'', \theta'') && \text{by SPM in } x \\
&\geq 0 && \text{by } x'' \in X^*(\theta'')
\end{aligned}$$

- All inequalities are therefore equalities
- Then $x' \vee x'' \in X^*(\theta')$ and $x' \wedge x'' \in X^*(\theta')$
- So $X^*(\theta)$ is increasing in the SSO.

Quasi-supermodularity

- $F\colon X \to \mathbb{R}$ is **quasi-supermodular** (QSPM) if $\forall x, x' \in X$:

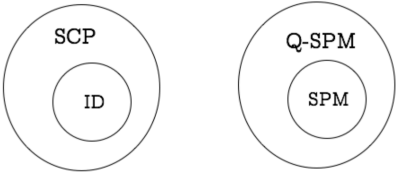
$$\begin{aligned} F(x) \geq F(x \wedge x') &\quad\Rightarrow\quad F(x \vee x') \geq F(x') \\ F(x) > F(x \wedge x') &\quad\Rightarrow\quad F(x \vee x') > F(x') \end{aligned}$$

- The contrapositive of each yields the equivalent:

$$\begin{aligned} F(x) < F(x \wedge x') &\quad\Leftarrow\quad F(x \vee x') < F(x') \\ F(x) \leq F(x \wedge x') &\quad\Leftarrow\quad F(x \vee x') \leq F(x') \end{aligned}$$

- If $F(x, \theta)$ is QSPM, then F obeys the **single crossing property** in (x, θ) if for all $x_2 \succ x_1$ and $\theta_2 \succ \theta_1$

$$\begin{aligned} F(x_2, \theta_1) \geq F(x_1, \theta_1) &\quad\Rightarrow\quad F(x_2, \theta_2) \geq F(x_1, \theta_2) \\ F(x_2, \theta_1) > F(x_1, \theta_1) &\quad\Rightarrow\quad F(x_2, \theta_2) > F(x_1, \theta_2) \end{aligned}$$



Ordinal Comparative Statics

- ***Milgrom-Shannon Theorem (1994):*** *Let X be a lattice, and Θ a poset. If $F\colon X \times \Theta \to \mathbb{R}$ obeys the SCP in (x, θ) and is QSPM in x, then $X^*(\theta)$ is monotone in the SSO.*
- Proof: Let $\theta' \succeq \theta$ with $x \in X^*(\theta)$ and $x' \in X^*(\theta')$.
- $x \vee x' \in X^*(\theta')$ since

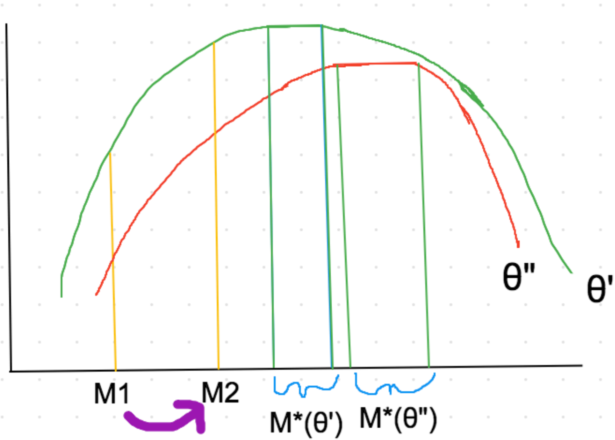
$$\begin{array}{rrcll} & F(x,\theta) & \geq & F(x \wedge x', \theta) & \text{by } x \in X^*(\theta) \\ \Rightarrow & F(x \vee x', \theta) & \geq & F(x', \theta) & \text{by QSPM} \\ \Rightarrow & F(x \vee x', \theta') & \geq & F(x', \theta') & \text{by SCP} \end{array}$$

- Next, $x \wedge x' \in X^*(\theta)$ since:

$$\begin{array}{rrcll} & F(x',\theta') & \geq & F(x \vee x', \theta') & \text{by } x' \in X^*(\theta') \\ \Rightarrow & F(x \wedge x', \theta') & \geq & F(x', \theta') & \text{by QSPM} \\ \Rightarrow & F(x \wedge x', \theta) & \geq & F(x', \theta) & \text{by SCP} \end{array}$$

- We applied the contrapositive forms of QSPM and SCP

Graphical Intuition for the Single Crossing Property



- Since the reals are a totally ordered set, any function on the reals is automatically SPM.

Ordinal Comparative Statics without a Lattice

- Let X and Θ be posets.
- The correspondence $\mathcal{X} : \Theta \to X$ is **nowhere decreasing** if $x_2 \in \mathcal{X}(\theta_1)$ and $x_1 \in \mathcal{X}(\theta_2)$ with $x_2 \succeq x_1$ and $\theta_2 \succeq \theta_1$ imply $x_1 \in \mathcal{X}(\theta_1)$ and $x_2 \in \mathcal{X}(\theta_2)$.
- So the correspondence does not fall anywhere
- ***Nowhere Decreasing Optimizers (2018):***
  Let X and Θ be posets. If $F: X \times \Theta \to \mathbb{R}$ obeys the single crossing property, then $X^*(\theta) \equiv \arg\max_{x \in X} F(x, \theta)$ is nowhere decreasing in θ.
- If $\theta_2 \succeq \theta_1$, $x_2 \in \mathcal{X}(\theta_1)$, $x_1 \in \mathcal{X}(\theta_2)$, and $x_2 \succeq x_1$, optimality and the single crossing property give $x_2 \in \mathcal{X}(\theta_2)$, since:

$$F(x_2, \theta_1) \geq F(x_1, \theta_1) \quad \Rightarrow \quad F(x_2, \theta_2) \geq F(x_1, \theta_2)$$

- Exercise: Prove that $x_1 \in \mathcal{X}(\theta_1)$

Monotone Stochastic Dominance

- Let X have cdf F and Y have cdf G.
- *First Order Stochastic Dominance:* $F \succsim_{FOD} G$ if $F(x) \leq G(x)\ \forall x$, iff survivors obey $\bar{F}(x) \geq \bar{G}(x)\ \forall x$
- ***Monotone Ranking Theorem.*** *If $F \succsim_{FOD} G$, then any mean of a monotone function is higher for X than Y.*
- Proof: Intuitively, every increasing function can be thought as the limit of the sum of step functions $\mathbb{I}\{x \geq a\}$.
- Partition the domain $[0,1]$ with $0 = a_0 < \cdots < a_N = 1$.
- Pick $w_0, w_1, \ldots, w_N > 0$.
- Define the weighted sum of step functions:

$$u_N(x) = \textstyle\sum_{k=0}^{N} w_k \mathbb{I}\{x \geq a_k\}$$

- The mean of $u(\cdot)$ is higher under F than G:

$$\mathbb{E}u_N(X) = \textstyle\sum_{k=0}^{N} w_k[1 - F(a_k)] \geq \sum_{k=0}^{N} w_k[1 - G(a_k)] = \mathbb{E}u_N(Y)$$

- Take the limit as the mesh vanishes $\Rightarrow Eu(X) \geq Eu(Y)$.
- Which utility function makes the ranking theorem iff?

Monotone Concave Stochastic Dominance

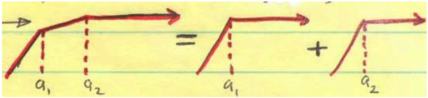
- *Second Order Stochastic Dominance:* $F \succsim_{SOSD} G$ if

$$\int_0^x F(t)dt \leq \int_0^x G(t)dt \quad \forall x$$

- ***Monotone Concave Stochastic Order.*** *If $F \succsim_{SOSD} G$, then any mean of a monotone concave utility function is higher for F than G.*
- Proof: We prove this for "ramp" functions $u_a = \min\{a, x\}$.
- Suppose that $0 \leq X, Y \leq M$. Then:

$$\begin{aligned} \mathbb{E}_F u_a(X) &= \int_0^a x dF(x) + \int_a^M a dF(x) \\ &= xF(x)\Big|_0^a - \int_0^a F(x)dx + a(1 - F(a)) \\ &= a - \int_0^a F(x)dx \end{aligned}$$

- So $\mathbb{E}_F u_a(X) \geq \mathbb{E}_G u_a(Y)$ iff $-\int_0^a F(x)dx \geq -\int_0^a G(x)dx$
- Intuitively, concave functions through the origin can be approximated as the limit of weighted sums of ramps
- So $\mathbb{E}_F u(\cdot) \geq \mathbb{E}_G u(\cdot)$.



- **Stochastic Dominance on Closed Cones**
- A **convex cone** is a vector space subset closed under positive linear combinations with positive coefficients.
- If the mean of F exceeds G on a set of functions V, then this holds on the convex cone $U = cc(V \cup \{\pm 1\})$.
- Example: If $U = \{$ concave functions$\}$ and $V = \{\min\langle 0, x-a\rangle\} \cup \{\pi(x) = -x\}$ then $U = cc(V \cup \pm 1)$

$$\mathbb{E}_F(\min\langle 0, X-a) \geq \mathbb{E}_F(\min\langle 0, X-a) \ \forall a \quad \text{and} \quad \mathbb{E}_F(-X) \geq \mathbb{E}_G(-X)$$

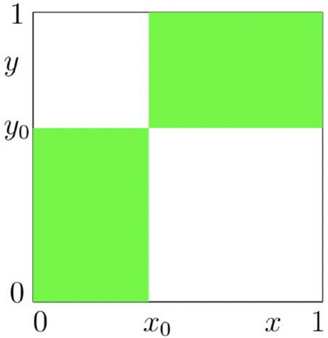
$\Rightarrow$ F, G have same mean $\Rightarrow \int_0^1 [1-F(t)]dt = \int_0^1 [1-G(t)]dt$

- *Mean Preserving Spread:* G is a MPS of F on $[0,1]$ if

$$\int_0^x F(t)dt \leq \int_0^x G(t)dt \quad \forall x, \text{ with equality at } x = 1$$

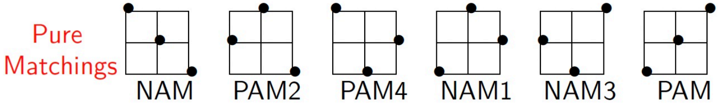
- ***Concave Stochastic Order.*** *If $F \succsim_{MPS} G$, then any mean of a concave utility function is higher for G than F.*
  Example. If F is a MPS of G, then $\sigma_F^2 \geq \sigma_G^2$.

PQD: Increased Sorting in Pairwise Matches



- *Positive quadrant dependence* (PQD) partially orders bivariate probability distributions $M \in \mathcal{M}(G, H)$
- *Sorting increases in the PQD order* if the mass in every northeast and southwest quadrant increases.
- So $M_2 \succeq_{PQD} M_1$ *iff* $M_2(x, y) \geq M_1(x, y)$ *for all* x, y
- We call M_2 *more sorted than* M_1

PQD Order is Not a Lattice: Three Type Example



$$\text{NAM} \preceq_{PQD} [\text{PAM2, PAM4}] \preceq_{PQD} [\text{NAM1, NAM3}] \preceq_{PQD} \text{PAM}$$

- Check that this is not a lattice!
  - For PAM2 or PAM4, each of NAM1, NAM3, and PAM are upper bounds, but there is no least upper bound
  - For NAM1 or NAM3, each of PAM2. PAM4, and NAM are lower bounds, but there is no greatest upper bound
  - Hence, PQD partial order is not a lattice on three types
  - Maybe we are missing mixed matchings that restore the lattice property. There is not.

PQD - SPM Stochastic Dominance Theorem

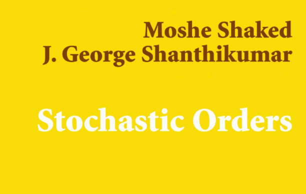
- This is missing from first year micro PhD curriculum:

Lemma (PQD Stochastic Dominance Theorem)

The PQD and SPM orders coincide on R^2, i.e. increases in the PQD order raise the total output for any SPM function f, and conversely:

$$M_2 \succeq_{PQD} M_1 \Leftrightarrow \int \phi(x,y) M_2(dx,dy) \geq \int \phi(x,y) M_1(dx,dy)$$

- Hence, PQD is called the supermodular order
- Method of cones intuition: a SPM function is in the cone of indicator functions $[x,\infty) \times [y,\infty) \cup (-\infty,x] \times (-\infty,y]$
- Homework: show that these indicator functions generate all SPM functions



Economics of the PQD Order

Lemma (PQD is Economically Relevant)

If sorting increases in the PQD order,
(a) the average distance between matched types falls;
(b) the covariance / correlation of matched pairs rises, and
(c) the coefficient in a linear regression of men's type on matched women's type increases.

- PROOF OF (*a*)
  - Use $\phi(x, y) = |x - y|^\gamma$ in PQD Stochastic Dominance Theorem
  - Claim: ϕ is SBM for all $\gamma \geq 1$ (homework: try to prove this!)

  $\Rightarrow$ $E[|X - Y|]$ over matched X, Y falls
- PROOF OF (*b*)
  - xy SPM $\Rightarrow$ covariance $E_M[XY] - E[X]E[Y]$ increases
  - Marginal distributions on X and Y are invariant to M.
  - $E[X^2]$ and $E[Y^2]$ fixed in match measure M

    $\Rightarrow$ correlation coefficient increases too
- PROOF OF (*c*): You try it! It's not hard!

Bayesian Information Economics

- Imagine we are trying to learning about the *state of the world* $\theta \in \Theta$ with a prior density $g(\theta)$ and cdf, if $\theta \in \mathbb{R}$
- Common binary state case: $\Theta = \{0, 1\}$, for states $\theta = L$ and $\theta = H$
- A signal is r.v. X whose density $f(x|\theta)$ depends on θ
- A *signal* is a family of r.v.s $\{f(x|\theta), \theta \in \Theta\}$, for every state
- Standard abuse of terminology: signal realization x is called the signal
- By Bayes rule, upon seeing x, the posterior density is

$$g(\theta|x) = \frac{pdf(\theta \text{ and } x)}{pdf(x)} = \frac{g(\theta)f(x|\theta)}{\int_{-\infty}^{\infty} f(x|s)g(s)} \propto g(\theta)f(x|\theta)$$

Odds Formulation of Bayes' Rule

- To eliminate the messy denominator, we often use odds
- Posterior odds = (prior odds) × (likelihood ratio)

$$\frac{g(\theta_2|x)}{g(\theta_1|x)} = \frac{g(\theta_2)f(x|\theta_2)}{g(\theta_1)f(x|\theta_1)}$$

- Example: A test to detect AIDS, whose prevalence is $\frac{1}{1000}$, has a false positive rate of 5%.
- Given a + result, what is the chance one is infected?
- Roughly: Posterior odds against infection are

$$\frac{999}{1} \times \frac{1}{19} \approx 1000/20 = 50$$

- One is infected with chance 2%

More Favorable Signals

- Signal realization x is *more favorable than* y if x leads to a stochastically higher posterior distribution on states than y: $G(\cdot|x) \succeq_{FSD} G(\cdot|y)$ for all nondegenerate priors G
- Idea: you to think the state θ is bigger after seeing x vs. y
- ***Theorem*** *(Milgrom, 1981): x is more favorable than y iff*

$$f(x|\theta_2)f(y|\theta_1) \geq f(x|\theta_1)f(y|\theta_2) \quad \forall \theta_2 > \theta_1 \qquad (\bigstar)$$

- Proof for binary signals θ_1, θ_2.
- For simplicity, assume equally likely atoms $g(\theta_1) = g(\theta_2)$
- x is more favorable than y only if posterior odds given

$$\theta_2 > \theta_1 \Leftrightarrow \frac{f(x|\theta_2)}{f(x|\theta_1)} \geq \frac{f(y|\theta_2)}{f(y|\theta_1)} \Leftrightarrow \frac{g(\theta_2|x)}{g(\theta_1|x)} \geq \frac{g(\theta_2|y)}{g(\theta_1|y)}$$

- Basically, on a two point support, likelihood ratio domination and FSD domination coincide

Proof of Milgrom's Theorem

- Proof: Assume positive signal densities at $x > y$.
- Be careful about dummy variables of integration!
- Inequality: $f(x|s)/f(x|\theta) \geq f(y|s)/f(y|\theta)$, if $\theta < \theta_2 < s$

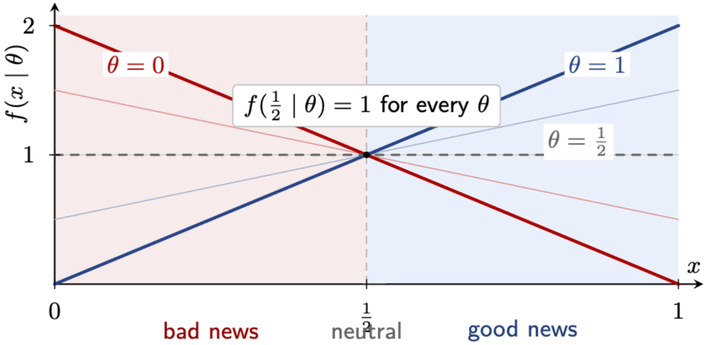
$$
\begin{aligned}
\frac{G(\theta_2|x)}{1 - G(\theta_2|x)} &= \frac{\int_{-\infty}^{\theta_2} f(x|s)dG(s)}{\int_{\theta_2}^{\infty} f(x|s)dG(s)} \\
&= \int_{-\infty}^{\theta_2} \frac{1}{\int_{\theta_2}^{\infty} [f(x|s)/f(x|\theta)]dG(s)} dG(\theta) \\
&\leq \int_{-\infty}^{\theta_2} \frac{1}{\int_{\theta_2}^{\infty} f(y|s)/f(y|\theta)dG(s)} dG(\theta) \\
&= \frac{G(\theta_2|y)}{1 - G(\theta_2|y)}
\end{aligned}
$$

Monotone Likelihood Ratio Property

- Signal $\{f(x|\theta)\}$ obeys the MLRP iff x is more favorable than any $y < x$
- $f(x,\theta)$ is *affiliated* if $f(x_1|\theta_1)f(x_2|\theta_2) \geq f(x_1|\theta_2)f(x_2|\theta_1)$
- Classic Signal Families with MLRP
  1. exponential: $f(x|\theta) = (1/\theta)e^{-x/\theta}$, $x \geq 0$
  2. skewed uniform: $f(x|\theta) = nx^{n-1}/\theta^n$, $0 \leq x \leq \theta$
  3. binomial: $f(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x = 0, 1, \ldots, n$
- Signal outcomes x and y are *equivalent* if $f(x|\theta_2)f(y|\theta_1) = f(x|\theta_1)f(y|\theta_2)\ \forall\theta_1, \theta_2$.
- Signal outcome x is *neutral news* if $f(x|\theta_1) = f(x|\theta_2)\ \forall\theta_1, \theta_2$ $\Rightarrow G(\cdot|x) \equiv G$
- Signal outcome x is *good news* if it is *more favorable than neutral news*, i.e. iff $f(x|\theta)$ is increasing in θ (*bad news* it if is less favorable).

Good News and Bad News

- Most signal distributions have no neutral news signal outcomes
- To see why neutral news is rare, consider this example: $\theta \in [0,1]$, and $f(x|\theta) = 2(\theta x + (1-\theta)(1-x))$.
- So $f(\frac{1}{2}|\theta) = 2(\frac{1}{2}\theta + \frac{1}{2}(1-\theta)) = 1$ for all states θ



Application: Contract Theory to Moral Hazard

- *Principal-Agent Problem (Holmstrom, 1979)*
- Agent expends effort θ, influencing stochastic profit π
- Profit π has density $f(\pi|\theta)$ given effort θ
- Agent's payoff to wealth w is $U(w)-\theta$, where $U' > 0 > U''$
- Principal has utility $V(\cdot)$, where $V' > 0 \geq V''$
- Principal and Agent are weakly/strictly risk-averse
- Optimal sharing rule: Principal gives the agent a profit share $s(\pi)$, where $\frac{V'(\pi - s(\pi))}{U'(s(\pi))} = b + c\frac{f_\theta(\pi|\theta)}{f(\pi|\theta)}$ $(c > 0)$
  - I won't prove this, but it solves the principal's optimization subject to agent obeying his IC constraints
- Proof that sharing rule rises when $f(\pi|\theta)$ has the MLRP:

$$\frac{f(\pi|\theta_2)}{f(\pi|\theta_1)} = \exp\left\{\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}[\log f(\pi|\theta)]d\theta\right\} = \exp\left\{\int_{\theta_1}^{\theta_2} \frac{f_\theta(\pi|\theta)}{f(\pi|\theta)}d\theta\right\}$$

- Intuition: Profits are a good signal of effort, so that $1 \geq s'(\pi) > 0$, if $\frac{f_\theta(\pi|\theta)}{f(\pi|\theta)}$ increases in π (the MLRP)

Logsupermodularity (LSPM)

- If $f > 0$, then f *logsupermodular* iff $\forall x_2 > x_1, \theta_2 > \theta_1$:

$$f(x_1|\theta_1)f(x_2|\theta_2) \geq f(x_1|\theta_2)f(x_2|\theta_1) \quad \Leftrightarrow$$

$$\log f(x_1|\theta_1) + \log f(x_2|\theta_2) \geq \log f(x_1|\theta_2) + \log f(x_2|\theta_1)$$

- Auction theorists call f *affiliated* iff f is logsupermodular
- Logsupermodularity on a lattice is defined without logs:

$$f(x)f(y) \leq f(x \vee y)f(x \wedge y)$$

- Multiplication preserves LSPM: f, g LSPM $\Rightarrow$ fg LSPM
- Addition need not preserve LSPM!
- An indicator function: $f(x, y) = 1$ if $x \geq y$ and 0 if $x < y$
- Prove that $\boxed{\text{indicator functions are LSPM}}$.

Signaling is Transitive

- Say X signals Y if $f(x|y)$ is LSPM
- If X signals Y (*unobserved*) and Y signals Z, does X signal Z?

Theorem

If $f(x, y)$ and $g(y, z)$ are LSPM, then so is $h(x, z) \equiv \int f(x, y)g(y, z)d\mu(y)$, for any positive measure μ.

- *LSPM is preserved under partial integration*
- Is this surprise? For addition does not preserve LSPM
- Milgrom and Weber (1982), "A Theory of Auctions and Competitive Bidding" (2020 Nobel Prize) repeatedly uses this property

Logsupermodularity, & the Preservation Lemma

- Ahlswede and Daykin (1979) proved the next result.

Theorem

$g(y,s)$ *LSPM* $\Rightarrow \int g(y,s)d\mu(s)$ *LSPM in* y

- Karlin and Rinott (1980) wonderfully proved it

Lemma (Four Functions Preservation)

Let $f_1, f_2, f_3, f_4 \geq 0$ *on* $\mathbb{R}^n$. *Then*

$$\boxed{PREMISE} \quad f_1(s)f_2(s') \leq f_3(s \vee s')f_4(s \wedge s') \quad \forall s, s' \in \mathbb{R}^n$$

$$\Longrightarrow$$

$$\int f_1(s)d\mu(s) \int f_2(s)d\mu(s) \leq \int f_3(s)d\mu(s) \int f_4(s)d\mu(s)$$

Proof of Four Functions Lemma (Easy Part)

- Use induction on the dimensionality of $\mathbb{R}^n$!
- We prove $n = 1$ case. The PREMISE with $s' = s$ gives:

$$f_1(s)f_2(s) \leq f_3(s)f_4(s) \tag{1}$$

- Since $\int f_i(s)\mu(ds) \int f_j(s)\mu(ds) = \int\int f_i(x)f_j(y)\mu(dx)\mu(dy)$, we need

$$\iint\limits_{x<y} [f_1(x)f_2(y)+f_1(y)f_2(x)]\,\mu(dx)\mu(dy) \leq \iint\limits_{x<y} [f_3(x)f_4(y)+f_3(y)f_4(x)]\,\mu(dx)\mu(dy)$$

- $a=f_1(x)f_2(y)$, $b=f_1(y)f_2(x)$, $c=f_3(x)f_4(y)$, $d=f_3(y)f_4(x)$
- It suffices to show that $a + b \leq c + d$.
- Claim 1: $d \geq a, b$.
  - Proof: $d = f_3(y)f_4(x) = f_3(x \vee y)f_4(x \wedge y)$ since $x < y$
  - PREMISE: $d \geq a$ by $s=x, s'=y$ & $d \geq b$ by $s=y, s'=x$.
- Claim 2: $ab \leq cd$.
  - Proof: multiply (1) at $s = x$ and $s = y$

Claims 1 & 2 $\Rightarrow (c + d) - (a + b) = [(d - a)(d - b) + (cd - ab)]\,/d \geq 0$

Partial Integration preserves LSPM

Theorem

$g(y,s)$ *LSPM* $\Rightarrow \int g(y,s)d\mu(s)$ *LSPM in* y

Proof:

- $f_1(s) = g(y,s)$, $f_2(s) = g(y',s)$, $f_3(s) = g(y \vee y', s)$, $f_4(s) = g(y \wedge y', s)$.
- Since g is LSPM, $f_1(s)f_2(s') \leq f_3(s \vee s')f_4(s \wedge s')$
- By the Four Functions Lemma,

$$\int f_1(s)d\mu(s) \int f_2(s)d\mu(s) \leq \int f_3(s)d\mu(s) \int f_4(s)d\mu(s)$$

- Unwrapping this, we get the desired inequality:

$$\int g(y,s)d\mu(s) \int g(y',s)d\mu(s) \leq \int g(y \vee y', s)d\mu(s) \int g(y \wedge y', s)d\mu(s)$$

Offline: Measure Inherits LSPM from Density

- $A \vee B \equiv \cup\{a \vee b,\ a \in A, b \in B\}$
- $A \wedge B \equiv \cup\{a \wedge b,\ a \in A, b \in B\}$
- Probability measure generated by f is $P(A) = \int_A f(s)ds$
- P is LSPM if $P(A \vee B)P(A \wedge B) \geq P(A)P(B)$
- ***Theorem**: If f is LSPM, then so is $P(A) = \int_A f(s)ds$*
- Proof: Let $f_1 = \mathbb{I}_A(x)$, $f_2 = \mathbb{I}_B(y)$, $f_3 = \mathbb{I}_{A\vee B}$, $f_4 = \mathbb{I}_{A\wedge B}$.
- The earlier $\boxed{\text{PREMISE}}$ holds for indicators, since when the left side of the inequality equals one, so does the right side:

$$\mathbb{I}_A = 1, \mathbb{I}_B = 1 \Rightarrow \mathbb{I}_{A\vee B} = 1 \text{ and } \mathbb{I}_{A\wedge B} = 1$$

  - Proof: $\mathbb{I}_A(x) = 1 \Leftrightarrow x \in A$, and $\mathbb{I}_B(y) = 1 \Leftrightarrow y \in B$.
  - But $x \in A$ and $y \in B \Rightarrow x \vee y \in A \vee B$, and $x \wedge y \in A \wedge B$.
- Set $f_1^* = f_1 f$, $f_2^* = f_2 f$, $f_3^* = f_3 f$, $f_4^* = f_4$.
- These obey the premise too! So by the Preservation Lemma:

$$P(A) = \int_A f_1^* \ \&\ P(B) = \int_B f_2^* \ \&\ P(A \vee B) = \int_{A\vee B} f_3^* \ \&\ P(A \wedge B) = \int_{A\wedge B} f_4^*$$

Logconcavity and LSPM

- $f > 0$ is *log concave* when $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$
- When $f > 0$ is C^2 on $\mathbb{R}$ when $\log f$ is concave, i.e.

$$(\log f)'' \leq 0 \Leftrightarrow (f'/f)' \leq 0 \Leftrightarrow ff'' \leq (f')^2$$

Lemma (Logconcavity and LSPM)

If $f > 0$ is log concave then $u(x,y) = f(y-x)$ is LSPM.

- Proof: $u_x = -f'(y-x)$, $u_{xy} = -f''(y-x)$, $u_y = f'(y-x)$.

$$u \text{ LSPM } \Leftrightarrow uu_{xy} \geq u_x u_y \Leftrightarrow -ff'' \geq -(f')^2 \Leftrightarrow f \text{ logconcave}$$

Logconcavity and Prekopa's Theorem (1973)

Theorem (Prekopa)

Let $H(x, y) : \mathbb{R}^{m+n} \to \mathbb{R}$ be log-concave:

$$H\left((1-\lambda)(x_1, y_1) + \lambda(x_2, y_2)\right) \geq H(x_1, y_1)^{1-\lambda} H(x_2, y_2)^{\lambda}$$

for $x_1, x_2 \in \mathbb{R}^m$ and $y_1, y_2 \in \mathbb{R}^n$ and $0 < \lambda < 1$.

Then its marginal $M(y) = \int_{\mathbb{R}^m} H(x, y)\, dx.$ is log-concave.

- ***Convolution Corollary***: *If f, g are logconcave on $\mathbb{R}$ then $h(x) \equiv \int_{-\infty}^{\infty} g(x-y) f(y) dy$ is logconcave.*
- Also, the cdf or survivor of a log-concave density is log-concave because the step function is log-concave: $g(x) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases}$
  1. Normal density: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ on $\mathbb{R}$
  2. Gamma density: $f(x) = \frac{\lambda^r x^{r-1}}{\Gamma(r)} e^{-\lambda x}$ is logconcave on $\mathbb{R}_+$ iff $r \geq 1$
  3. Beta density: $f(x) \propto x^{a-1}(1-x)^{b-1}$ is logconcave on $[0, 1]$ if $a, b \geq 1$, as with the uniform density

Logconcavity and Truncated Means

- Heckman and Honore (1990), Proposition 1
- Notation of Burdett (1996), which replicates this result without cite!
- Let $\underline{f_0}$ be a density, and $\underline{f_{k+1}}(z) \equiv \int_{-\infty}^{z} \underline{f_k}(x)dx$
- Left mean: $\underline{m}(z) = E[X|X \leq z] = \int_{-\infty}^{z} x f_0(x)dx / \underline{f_1}(z)$
- ***Proposition.*** *$\underline{m}'(z) \leq 1$ iff $\underline{f_2}$ is log-concave.*

$$
\begin{aligned}
\text{Proof: } \underline{m}'(z) &= \frac{\underline{f_1}(z) z f_0(z) - \underline{f_1}'(z) \int_{-\infty}^{z} x f_0(x)dx}{\left(\underline{f_1}(z)\right)^2} \\
&= \frac{\left(\int_{-\infty}^{z} f_0(x)dx\right) z f_0(z) - f_0(z) \int_{-\infty}^{z} x f_0(x)dx}{\left(\underline{f_1}(z)\right)^2} \\
&= f_0(z) \int_{-\infty}^{z} (z-x) f_0(x)dx / \left(\underline{f_1}(z)\right)^2 \\
&\equiv \underline{f_2}''(z)\underline{f_2}(z) / \left(\underline{f_2}'(z)\right)^2
\end{aligned}
$$

- This is ≤ 1 iff $\underline{f_2}''(z)\underline{f_2}(z) \leq \left(\underline{f_2}(z)\right)^2$, i.e. $\underline{f_2}$ is log-concave

Logconcavity and Truncated Expectations

- Logconcavity is often met and precludes jumps in expectations
- Left variance $\underline{V}(z) = Var(X|X \leq z) \Rightarrow \underline{V}'(z) \leq 1\ \forall z$ iff $\underline{f}_3$ log-concave
- Right mean $\overline{m}(z) = \mathbb{E}[X|X \geq z] \Rightarrow \overline{m}'(z) \leq 1\ \forall z$ iff $\bar{f}_2$ log-concave.
- Right variance $\overline{V}(z) = Var(X|X \leq z) \Rightarrow \overline{V}'(z) \leq 1\ \forall z$ iff $\bar{f}_3$ log-concave

Upcrossing Functions

- Let Θ be a poset. Then $\Upsilon : \Theta \to \mathbb{R}$ is *upcrossing* if
  - $\Upsilon(\theta) \geq 0 \Rightarrow \Upsilon(\theta') \geq 0$ for all $\theta' > \theta$
  - $\Upsilon(\theta) > 0 \Rightarrow \Upsilon(\theta') > 0$ for all $\theta' > \theta$
- Υ is *downcrossing* if $-\Upsilon$ is upcrossing
- Υ is *one-crossing* if it is upcrossing or downcrossing.

Upcrossing Preservation

- ***Karlin and Rubin (1956):*** *If Υ is upcrossing, and $\lambda > 0$ is nondecreasing, and μ is a measure, then*

$$\int_{-\infty}^{\infty} \Upsilon(s) d\mu(s) \geq (>) 0 \Rightarrow \int_{-\infty}^{\infty} \Upsilon(s)\lambda(s) d\mu(s) \geq (>) 0$$

- Proof: Let Υ first upcross at t_0. The right side equals

$$\begin{aligned} & \int_{-\infty}^{t_0} \Upsilon(s)\lambda(s) d\mu(s) + \int_{t_0}^{\infty} \Upsilon(s)\lambda(s) d\mu(s) \\ \geq (>) \quad & \lambda(t_0) \int_{-\infty}^{t_0} \Upsilon(s) d\mu(s) + \lambda(t_0) \int_{t_0}^{\infty} \Upsilon(s) d\mu(s) \\ = \quad & \lambda(t_0) \int_{-\infty}^{\infty} \Upsilon(s) d\mu(s) \geq 0 \end{aligned}$$

- Weakening the assumptions on Υ has led to key papers

Beyond Karlin and Rubin: Integral Single Crossing Property

- Often Υ is not upcrossing. Rather, only its expectation is upcrossing
- E.g. $E[V'(s)|x] \geq 0 \Rightarrow E[V'(s)|y] \geq 0$ if x is more favorable news than y

Corollary (Integral Single Crossing Property)

If $\lambda(x) \geq 0$ is nondecreasing, then (if all integrals are finite)

$$\int_{[y,\infty)\cap X} \Upsilon(x)dx \geq 0 \quad \textit{for all } y \qquad \Rightarrow \qquad \int_X \Upsilon(x)\lambda(x)dx \geq 0$$

Inequality is strict if $\int_X \Upsilon(x)dx > 0$ and $\exists m > 0$ s.t. $\lambda(x) \geq m$

- $\lambda \geq 0$ non$\downarrow$ $\Rightarrow \int_X \Upsilon(x)\lambda(x)dx$ lies in cone of $(\int_{[y,\infty)\cap X} \Upsilon(x)dx)\ \forall y$
- Proof: This is clear for sum of step functions λ
  - E.g.: $\lambda(x) = a_1\mathbb{I}_{x\geq x_1} + a_2\mathbb{I}_{x\geq x_2} \Rightarrow \int_X f(x)\lambda(x) = a_1\int_{x_1}\Upsilon(x) + a_2\int_{x_2}\Upsilon(x)$
- Offline Formal Proof (read on your own):
  - Since λ is monotone, its upper sets are $U = [y,\infty)$
  - Fix $M > 0$ very big. Let $\lambda_M = M$ for $x \in U(M)$ and $\lambda_M(x) = \lambda(x)$ if not
  - Banks Lemma ($m > 0$ on next slide) $\Rightarrow \int_X \Upsilon(x)\lambda_M(x)dx \geq 0$
  - If $M \uparrow \infty$, get $\int_X \Upsilon(x)\lambda(x)dx \geq 0$ by monotone convergence theorem

Offline: Dallas Banks Integral Inequality

- Beesack (1957), "A note on an integral inequality"
- *upper set* $U(y) = \{x \in X \subset \mathbb{R}, \lambda(x) \geq y\}$ of function λ

Lemma (Banks Lemma, 1963)

If $m \leq \lambda(x) \leq M < \infty \; \forall x \in X$ *then*

$$\int_X \Upsilon(x)\lambda(x)dx = m\int_X f(x)dx + \int_m^M \left(\int_{U(y)} f(x)dx\right) dy \quad (\dagger)$$

- See "layer cake" integral notion in wikipedia, but swap $\lambda(x)$ and $f(x)$
- Proof: Define $F(y) = \int_{U(y)} f(x)dx$ for $y \in [m, M)$, and $F(M) = 0$
- Layer Cake Claim: $\int_X f(x)\lambda(x)dx = -\int_m ydF(y)$
  - Proof: Take a partition $m = y_0 < y_1 < \cdots < y_n = M$
  - $\int_m ydF(y) \sim \sum_{k=1}^n y_i[F(y_{k-1}) - F(y_k)] \sim \sum_{k=1}^n \lambda(x_k)f(x_k)\Delta x_k$ since $y_k \leq \lambda(x) \leq y_{k+1}$ on $U(y_{k-1}) \setminus U(y_k)$
- Integrate Layer Cake Claim by parts to get $(\dagger)$

$$\int_X f(x)\lambda(x)dx = -yF(y)|_m^M + \int_m^M F(y)dy$$

Single Crossing Property

- Let $X \subseteq \mathbb{R}$ and $f, g : X \to \mathbb{R}$.
- g dominates f by the *single crossing property* (or $g \succsim_{SC} f$) if

$$\forall x'' > x' : \qquad f(x'') - f(x') \geq (>)\, 0 \quad \Longrightarrow \quad g(x'') - g(x') \geq (>)\, 0$$

- Since $\mathbb{R}$ is a chain, Milgrom–Shannon (1994) imply that

$$\arg\max_{x \in Y} g(x) \sqsupseteq \arg\max_{x \in Y} f(x) \quad \text{for every } Y \subseteq X$$

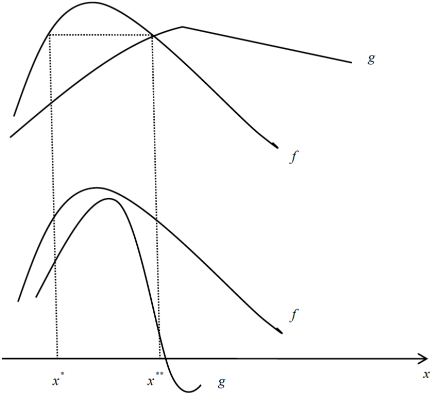
if and only if $g \succsim_{SC} f$

Interval Dominance Order

- The single crossing property is often too strong:
  - $\arg\max g$ can be right of $\arg\max f$ on an interval when $g \succsim_{SC} f$ fails.
- g dominates f by the *interval dominance order* (or $g \succsim_I f$), if

$$f(x^{**}) - f(x^{*}) \geq (>)\, 0 \quad \Longrightarrow \quad g(x^{**}) - g(x^{*}) \geq (>)\, 0.$$

for every pair $x^{**} > x^{*}$ such that $f(x^{**}) \geq f(x)$ for all $x \in [x^{*}, x^{**}]$



Quah and Strulovici (2009)

- A subset $I \subseteq X$ is an *interval of* X if for any $x', x'' \in I$ and any $x \in X$ such that $x' < x < x''$, we also have $x \in I$.
- This is more general than real interval notion: If $X = \{1, 2, 3, 5, 6\}$ then $\{1, 2\}$ and $\{3, 5, 6\}$ are intervals, but not $\{3, 6\}$.
- **Theorem** (Quah and Strulovici (2009)) For real-valued f, g on $X \subset \mathbb{R}$,

$$g \succsim_I f \iff \arg\max_{x \in I} g(x) \geq \arg\max_{x \in I} f(x) \quad \text{for every interval } I \subseteq X.$$

- The IFF Milgrom and Shannon single crossing theorem is:

$$g \succsim_{SC} f \iff \arg\max_{x \in S} g(x) \geq \arg\max_{x \in S} f(x) \quad \text{for every subset } S \subseteq X.$$

A Simple Way to Apply the Interval Dominance Order

- Proposition 2 in QS (2009) relax Milgrom and Shannon's SCP premise

Theorem

The maximizer set $\arg\max_x V(x,t)$ *increases in the SSO in t provided:*

$$\exists \lambda > 0 \text{ nondecreasing: } V_x(x|t_2) \geq \lambda(x) V_x(x|t_1) \ \forall t_2 > t_1$$

- Novel proof by my former advisee Jianrong Tian using cone logic:
- Let $t_2 > t_1$ and $x_i \in \arg\max_x V(x|t_i)$ for $i = 1, 2$
- **Claim 1**: $\max(x_1, x_2) \in \arg\max_x V(x|t_2)$
- True if $x_2 \geq x_1$. Assume $x_1 > x_2$.

$$V(x_1|t_2) - V(x_2|t_2) = \int_{x_2}^{x_1} V_x(x|t_2)dx \geq \int_{x_2}^{x_1} \lambda(x) V_x(x|t_1)dx \quad (\ddagger)$$

- $x_1 \in \arg\max_x V(x, t_1) \Rightarrow \int_y^{x_1} V_x(x|t_1)dx \geq 0 \ \forall y \in [x_2, x_1]$.

$\Rightarrow$ $\int_{x_2}^{x_1} \lambda(x) V_x(x|t_1)dx \geq 0$ by integral SCP

- By $(\ddagger)$, $V(x_1|t_2) \geq V(x_2|t_2)$
- Altogether, $\max(x_1, x_2) \in \arg\max V(x, t_2)$

Topkis without the Single Crossing Property

- **Claim 2**: $\min(x_1, x_2) \in \arg\max V(x|t_1)$.
- True if $x_1 \leq x_2$. Assume $x_1 > x_2$.
- For a contradiction, assume that $V(x_1|t_1) > V(x_2|t_1)$.
- Then $\int_{x_2}^{x_1} V_x(x|t_1)dx > 0$.
- $x_1 \in \arg\max V(x, t_1) \Rightarrow \int_{y}^{x_1} V_x(x|t_1)dx \geq 0 \ \forall y \in [x_2, x_1]$.

$\Rightarrow$ $\int_{x_2}^{x_1} \alpha(x) V_x(x|t_1)dx > 0$ by strict integral SCP

- By (‡), $V(x_1|t_2) - V(x_2|t_2) > 0$
- This contradicts $x_2 \in \arg\max V(x|t_2)$.

$\Rightarrow$ $V(x_1|t_1) = V(x_2|t_1)$

$\Rightarrow$ $\min(x_1, x_2) \in \arg\max V(x|t_1)$.

- PS: The proof Quah and Strulovici is not as elegant or short

Application to Cost-Benefit Analysis

- Choose x in interval X to maximize differentiable benefits minus costs:

$$\max_{x\in X} \Pi(x), \qquad \Pi(x) = B(x) - C(x), \qquad \widetilde{\Pi}(x) = \widetilde{B}(x) - \widetilde{C}(x).$$

- Let $\lambda > 0$ be nondecreasing.
- Assume $\widetilde{B}'(x) \geq \lambda(x)B'(x)$ and $\widetilde{C}'(x) \leq \lambda(x)C'(x)$ for all $x \in X$,
- Then $\arg\max_{x\in X} \widetilde{\Pi}(x) \sqsupseteq \arg\max_{x\in X} \Pi(x)$
- Proof: Since $\widetilde{\Pi}'(x) = \widetilde{B}'(x) - \widetilde{C}'(x) \geq \lambda[B'(x) - C'(x)] = \lambda(x)\Pi'(x)$

$\Rightarrow$ $\widetilde{\Pi} \succeq_I \Pi$. Apply the Interval Dominance Order comparative static.

Total Positivity (Karlin, 1968)

- Total positivity generalizes logsupermodularity, but only on the reals
- $u : A \times B \to \mathbb{R}$ is *totally positive of order* k (TP_k if $\forall m = 1, \ldots, k$ and $x_1 < \cdots < x_m$ in $A \subseteq \mathbb{R}$ and $y_1 < \cdots < y_m$ in $B \subseteq \mathbb{R}$ ($\Longleftarrow$ scalar!)

$$\det \begin{bmatrix} u(x_1, y_1) & \cdots & u(x_1, y_m) \\ \vdots & & \vdots \\ u(x_m, y_1) & \cdots & u(x_m, y_m) \end{bmatrix} \geq 0$$

- TP_1 means nonnegative, and TP_2 is LSPM on $\mathbb{R}^2$
- Easily, $TP_k \Rightarrow TP_{k'}$ $\forall k' \leq k$.
- $u(x, y)$ is TP (or *totally positive*) if it is TP_k $\forall k < \infty$.
- *Lemma: If $v, w \geq 0$ on A and B, and $u(x, y)$ is TP_k, then $v(x)w(y)u(x, y)$ is TP_k on $A \times B$.*
- *Lemma: If v and w are comonotone, and f is TP_k on $A \times B$, then $u(v(x), w(y))$ is TP_k on $A \times B$.*
  1. $u(x, y) = e^{xy}$ is TP $\Rightarrow e^{-(x-y)^2} = e^{-x^2} e^{-y^2} e^{2xy}$ is TP
  2. $u(x, y) = \frac{1}{x+y}$ is TP
  3. $u(x, y) = C(x, y)$ is TP

Variation Diminishing Property (VDP)

- *TP* preserves monotonicity and convexity.

Theorem (Monotonicity Preservation)

Let $\int f(x,y)d\mu(y) = 1\ \forall x$. If f is TP_2 and $w(y)$ is monotonic, then $u(x) = \int f(x,y)w(y)d\mu(y)$ is co-monotonic with w.

- Applications: When $f(x,y)\mu(y)$ is a probability density over random outcomes y given x, where $f(x,y)$ is the MLRP signal density
- Proof: w monotonic $\Leftrightarrow$ $w(y) - c$ is upcrossing $\forall c \in \mathbb{R}$
- Since $\int f(x,y)d\mu(y) = 1$, for any $\mu \in \mathbb{R}$,

$$u(x) - c = \int f(x,y)(w(y) - c)d\mu(y)$$

- If $w(y) - c$ changes sign $-$ to $+$, then so does $u(x) - c$ by Karlin and Rubin (1956) Upcrossing Preservation, since $f(x,y)$ is LSPM. $\square$

Karlin's Variation Diminishing Property (VDP)

- Let $S(f)$ be the supremum number of sign changes in $f(t_2) - f(t_1), \ldots, f(t_k) - f(t_{k-1})$ across all sets $t_1 < \cdots < t_k$.
- For a function $w(y)$, define $u(x) \equiv \int f(x, y)w(y)d\mu(y)$.

Theorem (Variation Diminishing Property, Karlin (1968))

Let $f(x, y)$ be TP_k. If $S(w) \leq k - 1$, then $S(u) \leq S(w)$, and u and w have the same arrangement of signs (left to right) in the domain.

- Proof: Obvious for $k = 1$; proven already for $k = 2$.
- For $k > 2$, Karlin's proof is a mess.
- Read Andrea Wilson's cool induction proof!
  - Induction step: if $\sum_y f(x, y)w(y)$ is n-crossing, initially $+$ to $-$, and f is TP_{n+1}, then $w(y)$ is n-crossing with an initial downcrossing
  - Let $x_1 < \cdots < x_{n+1}$ and $c_1, \ldots, c_n$ with $(-1)^{j+1}c_j > 0$ with

$$\sum_{i=1}^{n} f(x_j, y_i)w(y_i) = c_j \quad \text{for} \quad j = 1, 2, \ldots, n+1$$

  - She uses Cramer's rule: The TP Determinants are key